Time-optimal control of a swing

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Introduction

For several decades now, a steady stream of research in the field of biomechanics has supported the idea that humans while performing skilled tasks act as self-optimizing machines [1–4]. Accordingly, skilled human motion is viewed as an optimization problem entailing the minimization of some cost function with constraints on the control inputs and the phase-space. The set of constraints is generally nonlinear and includes magnitude constraints on the control inputs corresponding to the maximum forces and moments that the muscles can apply and constraints on the phase-space corresponding to the maximum displacements and rotations that are permitted by the joints, muscles, and other passive structures. While the constraints are determined by experimental and analytical means, researchers are often in disagreement over the cost function. The cost function that is minimized is believed to vary from activity to activity and there is at times some disagreement about the cost function even while considering the same activity [5].

As is to be expected, human locomotion is the subject matter of much research, such as [6], which tries to determine if the preferred frequency of locomotion is predicted by assuming a driven harmonic oscillator model for the lower human body and minimizing the energy required to drive the oscillator. Other forms of human motion such as rowing, swimming, and jumping have also been studied. For example [7] deals with the optimization of muscle coordination for maximal jumping.

Most of the work cited above is based on models where the human body is considered as a
collection of appendages and joints. These complex models often preclude analytical solutions to the optimization problem forcing researchers to turn to numerical methods. Here however we consider a rider on the common playground swing, a system which can be modeled adequately by a rather simple model. A look around the neighborhood playground reveals that children playing on a swing all follow a similar strategy, they stand up and at the lowest point of their motion and squat at the highest. In this paper we show that in the limit that the process of standing and squatting is carried out instantaneously, this pumping strategy adopted by children is time optimal.

**Pumping of a Swing**

The analysis of the pumping of a swing is a classical problem and has been addressed by several researchers from many different perspectives. Different pumping strategies (standing [8] and sitting [9]) have been considered, and the rider has been modeled at times as a point mass [10], at times as a rigid body with a moment of inertia [11], and at times as an assembly of linked dumbbells [12]. In this paper we restrict our attention to the problem of time-optimal pumping of a swing, where the rider is allowed to pump the swing by alternately standing and squatting.

The related problem of vertically stabilizing a pendulum of fixed length attached to a moving cart has a rich literature. For example, [13] presents a solution by regulating the swing energy and providing internal stability by regulating the cart position. The real-time time-optimal control problem for the cart-and-pendulum system is considered in [14]. A system analogous to the variable-length pendulum is the Acrobot, which is a two-linked pendulum actuated at the ‘elbow-joint’ but not at the ‘shoulder-joint.’ The general swing-up control problem for the Acrobot is discussed in [15], while the minimum-time problem for the Acrobot is considered in [16].
Equations of motion

The rider-and-swing system is modeled as a pendulum of mass $m$ with a variable length $l(t)$. The rope is taken to be massless, and the angle that it makes with the vertical is denoted by $\theta(t)$. Let $l_+, l_-$, and $L$ denote the maximum, minimum, and mean length of the pendulum respectively. The squatting and standing length is therefore denoted by $l_+$ and $l_-$ respectively. Let $\Delta l = l_+ - l_-$. Dissipative forces such as bearing-friction and wind-drag are ignored. Note that under the current formulation the length of the pendulum is the control input for the system.

Conservation of angular momentum for the pendulum bob gives

$$\frac{dH}{dt} = \tau,$$

where $H$ is the angular momentum of the bob about the fixed support, and $\tau$ is the net torque about the fixed support due to all the forces acting on the point mass. Therefore,

$$\frac{d}{dt}(l(t)^2\dot{\theta}(t)) = -g l(t) \sin(\theta(t)). \quad (1)$$

After differentiating and rearranging the terms we obtain,

$$\ddot{\theta} + \frac{2i\dot{\theta}}{l} + \frac{g \sin(\theta)}{l} = 0, \quad (2)$$

where dependence of $l$ and $\theta$ on $t$ has been dropped for succinctness.

Periodic variation of the length

The analysis of pumping of a swing is now addressed from several different perspectives. First we discuss how, under certain conditions, periodic variations in the length of the swing
results in motion of increasing amplitude [17]. Making the transformation $\nu = l \theta$, and using $\sin(\theta) \approx \theta$ for small $\theta$ we have,

$$\ddot{\nu} + \frac{1}{l} (g - \ddot{l}) \nu = 0.$$ 

Letting the length of the pendulum vary sinusoidally, $l(t) = L(1 + \epsilon \cos(\omega t))$, where $0 < \epsilon \ll 1$, and scaling time such that $\tau = \omega t$ we have,

$$\nu'' + \frac{(\delta + \epsilon \cos(\tau))}{(1 + \epsilon \cos(\tau))} \nu = 0,$$

where $(\cdot)'$ denotes $d(\cdot)/d\tau$ and $\delta = \frac{g/L}{2\pi^2}$. As a first-order approximation in $\epsilon$ we obtain Mathieu’s equation,

$$\nu'' + (\delta + \epsilon (1 - \delta) \cos(\tau)) \nu = 0.$$ 

This equation arises in many different mathematical and engineering contexts and its properties have been studied in detail [18]. For certain values of $\delta$ and $\epsilon$ the solution to this equation is unbounded. One such solution for $\delta = 1/4$ and $\epsilon = 0.1$ is shown in Figure 1. Note that this value of $\delta$ corresponds to the situation when the frequency of variation of the length of the pendulum is twice the pendulum’s natural frequency.

**Instantaneous variation of the length**

While thus far the length has been taken to be a continuous function of time, the application of control theory is facilitated by allowing the length to be a piecewise continuous function. To build intuition about the problem we now use physical reasoning and elementary calculus to explain the process of pumping of a swing when the length is allowed to vary instantaneously [19].

Assume the rider is initially squatting and moving toward the mid-point of her trajectory, corresponding to $\theta = 0$ (Top of Figure 2). Let $t_0$ denote the time when the rider is approaching the mid-point and $\theta \approx 0$, and $t_0 + \Delta t$ denote the time when she is travelling away from the mid-point and is completely standing up, where $|\theta| \leq \epsilon$ for $t \in [t_0, t_0 + \Delta t]$. Using the fact that
the change in angular momentum is obtained by integrating the torque with respect to time,

\[ l^2 \dot{\theta}_{\text{stand}} - l^2_+ \dot{\theta}_{\text{squat}} = - \int_{t_0}^{t_0+\Delta t} g l(t) \sin(\theta(t)) dt. \] (3)

Since \(|\sin(\theta(t))| \leq \epsilon\) during the entire maneuver the right-hand-side of (3) is \(O(\epsilon)\), and as \(\epsilon \to 0\) we have

\[ \dot{\theta}_{\text{stand}} = \left( \frac{l_+}{l_-} \right)^2 \dot{\theta}_{\text{squat}}. \]

Since \(l_+ > l_-\), the angular velocity of the rider increases by this action and she attains a larger amplitude of oscillation. The rider thus increases the amplitude of oscillation by standing up at the mid-point of the motion. To be able to impart this jump to the angular velocity repeatedly the rider must be able to return to the squatting position at some other point along the motion, ideally without affecting the velocity or the angle.

Consider then the point in the swing’s motion where \(\dot{\theta} \approx 0\), corresponding to the highest-point in the rider’s path. Let the rider be standing at a time \(t_1\) as she approaches the highest point and return to the squatting position at time \(t_1+\Delta t\) as she recedes from the highest point, where \(|\dot{\theta}| \leq \epsilon\) for \(t \in [t_1, t_1+\Delta t]\) (The right side of Figure 2). Then since the difference in the value of \(\dot{\theta}\) before and after squatting does not differ by more than \(2\epsilon\), in the limit as \(\epsilon \to 0\) the angular velocity remain unaffected by this maneuver. Also if the process is carried out instantaneously our intuition dictates that the angle should also remain unaffected. To verify our intuition we integrate (1) from \(t_1\) to a time \(t\), where \(t \leq t_1 + \Delta t\), obtaining

\[ l^2(t) \dot{\theta}(t) - l^2(t_1) \dot{\theta}(t_1) = - \int_{t_1}^{t} g l(t) \sin(\theta(t)) dt. \]

Dividing by \(l^2(t)\) and integrating from \(t_1\) to \(t_1+\Delta t\),

\[ \theta(t_1 + \Delta t) - \theta(t_1) - \int_{t_1}^{t_1+\Delta t} \left( \frac{l_+}{l(t)} \right)^2 \dot{\theta}(t_1) dt = \int_{t_1}^{t_1+\Delta t} - \frac{1}{l^2(t)} \int_{t_1}^{t} g l(\tau) \sin(\theta(\tau)) d\tau dt. \]

In the limit that the change in length is carried out instantaneously when the angular velocity
is zero, that is $\epsilon \rightarrow 0$ and $\Delta t \rightarrow 0$ simultaneously, both the above integrals tend to zero. The angle is also therefore unaffected during this maneuver and our intuition is substantiated.

Thus by standing and squatting at the right times during the motion, the rider increases the amplitude of oscillation of the swing. When the process is carried out in reverse, interchanging the squatting and standing maneuvers, the amplitude of oscillation of the pendulum is decreased. The amplitude of oscillation is thus increased or decreased by suitably altering the length of the pendulum.

**Energy Considerations**

Assuming bounded motion ($\theta_{max} \leq \pi$), this rather simple pumping strategy leads to an exponential increase in the mechanical energy of the system [8]. Note that the pumping strategy outlined above and captured in Figure 2 is the one in which the rider does the most work per oscillation, since the rider stands up at the lowest point, doing work against both the gravitational and centrifugal forces, and squats at the highest point, where the centrifugal force vanishes and the component of the gravitational force along the length of the swing is the least. Since all the work done is converted into stored energy, an increase in the mechanical energy of the system results in a corresponding increase in the amplitude of oscillation. To demonstrate the exponential increase in the mechanical energy, consider the situation where the rider is approaching the mid-point of her trajectory while squatting, and is undergoing motion whose amplitude of oscillation and maximum speed is given by $\theta_{max}$ and $v_{max}$ respectively. The total energy of the system is then given by

$$E = \frac{1}{2}mv_{max}^2 = mgl(1 - \cos \theta_{max}).$$
The work done at $\theta = 0$ when the rider stands up instantaneously is

$$W = (mg + \frac{mv_{max}^2}{l_+})\Delta l,$$

and the new total energy of the system $E'$ is given by

$$E' = E + W = mg\Delta l + E(1 + 2\frac{\Delta l}{l_+}).$$

The work done by the rider at the top of the swing’s motion when she returns to the squatting position is

$$W' = -mg\Delta l \cos(\theta'_{max}),$$

where $\theta'_{max}$ is the new amplitude. The total energy of the system $E''$ is then given by

$$E'' = E' + W' \approx E(1 + 3\frac{\Delta l}{l_+}),$$

where higher-powers of $\frac{\Delta l}{l_+}$ have been ignored. Since this process is carried out twice per oscillation the total energy at the end of $n$ oscillations $E_n$ is

$$E_n = E(1 + 3\frac{\Delta l}{l_+})^{2n}.$$

Thus energy considerations also indicate how the amplitude of oscillation of the swing can be increased by suitably changing the length.

**Optimal Control: Linear Case**

**Linearized System Equations**

In the previous section we analyzed the problem of pumping of a swing from different perspectives, and we now turn our attention to the related optimal control problem. The
linearized problem is treated first in some detail, and we show using the celebrated minimum principle that the pumping strategy discussed earlier is a time-optimal strategy. A broad outline to the solution of the nonlinear problem is presented in the next section.

Rewriting (2) as two first-order differential equations

\[
\begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= -\frac{2lz_2}{l} - \frac{g\sin(z_1)}{l}.
\end{align*}
\] (4)

The system (4) is an impulsive control system, which is denoted in a more general form by \( \dot{x} = \Phi(t, x, u, \dot{u}) \), with the initial condition \( x(0) = \bar{x} \) and where \( u \) is the control input. For such systems, when the control input \( u \) is discontinuous the corresponding trajectory may be discontinuous as well. Thus the solution to (4) defined in the usual Caratheodory sense, that is an absolutely continuous function that satisfies (4) at almost all \( t \), is no longer applicable. However under certain conditions [20,21] a non-impulsive system is obtained, such that solutions to the non-impulsive system are representations of the solution to the original impulsive system. In the present case such a system is obtained by making the nonlinear substitution \( x_2 = z_2l^2 \) (the angular momentum!). We then have

\[ \dot{x}_2 = -gl\sin(z_1). \]

Rewriting \( z_1 \) by \( x_1 \) in (4) we have the following set of equations in the variables, \( x_1 \) and \( x_2 \), that capture the dynamics of our system,

\[
\begin{align*}
\dot{x}_1 &= \frac{x_2}{l^2}, \\
\dot{x}_2 &= -gl\sin(x_1).
\end{align*}
\] (5)

Note that derivatives of \( l \) no longer appear in the system equation. Making the approximation, \( l(t) = L(1 + \epsilon u(t)) \), where \( 0 < \epsilon \ll 1 \) and \( |u(t)| \leq 1 \), and after expanding and retaining only
the first order terms we have

\[
\begin{align*}
\dot{x}_1 &= \frac{x_2}{L^2} - \frac{2x_2\epsilon u}{L^2}, \\
\dot{x}_2 &= -gL \sin(x_1) - gL \sin(x_1)\epsilon u,
\end{align*}
\]

(6)

where the dependence of \(x_1, x_2,\) and \(u\) on \(t\) is not stated explicitly for succinctness. Note that

the squatting position with length \(l_+\) corresponds to \(u = +1,\) and the standing position with

length \(l_-\) corresponds to \(u = -1.\) Linearizing about the origin and without any loss of generality

setting \(L = 1\) and \(g = 1,\) we obtain

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - u
\begin{bmatrix}
0 & 2\epsilon \\
\epsilon & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}. 
\]

(7)

Rewriting in matrix form we have

\[
\dot{x} = Ax + uBu.
\]

Systems of this form which are linear in \(x\) and \(u\) separately, but jointly quadratic, are known

as bilinear systems [22,23]. A detailed discussion on controllability and reachability for bilinear

systems is found in [23]. Here we note that the system is not completely controllable, that is,

it is not possible to drive the system from any initial state to any prescribed terminal state

in finite time with admissible controls. Specifically, if the system is at the origin then the

system does not leave that state for any admissible control. The physical implication is that

one needs to give a push to the swing as one starts swinging, the push being an input which

is not admissible according to our problem definition. Also the problem of reaching the origin

is not well-posed as the origin cannot be reached in finite time. With these points in mind

we consider the problem of bringing the swing from an initial non-zero angle and zero angular

velocity, \(x_{ini} = [\bar{x}_1, 0]^T,\) to a target circle of radius \(\rho\) centered about the origin, in minimum

time. The control input is bounded in magnitude such that \(|u(t)| \leq 1.\)
The Pontryagin Minimum Principle

Having obtained the equations of motion and stated the problem that we aim to solve, we now turn to the theoretical tools that are employed in the solution. Traditionally the calculus of variations approach was employed to tackle such optimization problems. This methodology however suffers from the disadvantage that it cannot be applied to problems where the control input is bounded, a class of problems that is of great engineering significance. The minimum principle overcomes this limitation and at the same time states the necessary conditions in an extremely convenient form, resulting in its wide-applicability to many interesting and challenging problems.

The theory of optimal control is not discussed in any great detail other than to state the main result in a form that is applicable to our problem. The interested reader is guided to the excellent references [24–27] for a more in-depth introduction to the field of optimal control. The existence of an optimal control for our problem is guaranteed in [28], and since the time-optimal control problem is normal, we conclude that if a time-optimal control exists for our problem then it is unique.

The minimum principle presupposes the existence of an optimal control input $u^*(t)$ and states the necessary conditions that have to be satisfied. Let $x^*(t)$ be the trajectory corresponding to $u^*(t)$ which satisfies the boundary conditions, that is $x^*(0) = x_{ini} = [\bar{x}_1, 0]^T$ and $x^*(T^*) \in S$, where $S = \{x : ||x|| = \rho\}$, and $T^*$ is the optimal time. The Hamiltonian $\mathcal{H}$ is given by the relation

$$\mathcal{H}(x(t), p(t), u(t)) = 1 + p(t)^T[Ax(t) + u(t)Bx(t)],$$

where the superscript $T$ denotes the transpose. Then there exists a corresponding nontrivial costate vector $p^*(t)$ such that;
1. The following differential equations are satisfied by the state and the costate vectors,

\[ \dot{x}^*(t) = \frac{\partial H}{\partial p}(x^*(t), p^*(t), u^*(t)), \]

\[ \dot{p}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), p^*(t), u^*(t)), \]

with boundary conditions \( x^*(0) = x_{ini} \) and \( x^*(T^*) \in S \).

2. For all \( t \in [0, T^*] \), the Hamiltonian is minimized by the optimal control input,

\[ H(x^*(t), p^*(t), u^*(t)) = \min_{|u(t)| \leq 1} H(x^*(t), p^*(t), u(t)). \]

3. The value of the Hamiltonian for the optimal control input is zero,

\[ H(x^*(t), p^*(t), u^*(t)) = 0, \quad t \in [0, T^*]. \]

4. The costate vector \( p^*(T^*) \) is transversal to \( S \), that is

\[ p^*(T^*)^T [x - x^*(T^*)] = 0, \quad x \in M[x^*(T^*)], \]

where \( M[x^*(T^*)] \) is the tangent plane of \( S \) at \( x^*(T^*) \).

**Bang-Bang Control**

Using (10) the optimal control is given by

\[ u^*(t) = \text{sgn}[2p_1^*(t)x_2^*(t) + p_2^*(t)x_1^*(t)]. \]

Since the expression inside the \text{sgn} function vanishes only at isolated points, the optimal control is bang-bang, that is it is piecewise constant and takes on values of either +1 or -1. Let
\( \bar{u} = \{ \pm 1 \} \). Therefore \( u^*(t) \in \bar{u} \).

**State and Costate vectors**

We have from (8)

\[
\begin{bmatrix}
\dot{x}_1^*
\dot{x}_2^*
\end{bmatrix} = \begin{bmatrix}
0 & (1 - 2\epsilon \bar{u}) \\
-(1 + \epsilon \bar{u}) & 0
\end{bmatrix} \begin{bmatrix}
x_1^*
 x_2^*
\end{bmatrix},
\]

with the initial condition \( x^*(0) = x_{ini} \), and where with a slight abuse of notation, \( \bar{u} \) is a piecewise constant function taking a value of either +1 or -1 as noted earlier. Rewriting as a single second-order differential equation

\[
\ddot{x}_1^*(t) + \omega^2 x_1^*(t) = 0,
\]

where \( \omega = \sqrt{1 - \epsilon \bar{u} - 2\epsilon^2 \bar{u}^2} \). We then obtain,

\[
\begin{align*}
x_1^*(t) &= \bar{x}_1 \cos(\omega t), \\
x_2^*(t) &= -\frac{\omega \bar{x}_1}{1 - 2\epsilon \bar{u}} \sin(\omega t). \\
\end{align*}
\tag{14}
\]

Let us now turn our attention to the costate vector. From (9) we have

\[
\begin{bmatrix}
p_1^*
p_2^*
\end{bmatrix} = \begin{bmatrix}
0 & (1 + \epsilon \bar{u}) \\
-(1 - 2\epsilon \bar{u}) & 0
\end{bmatrix} \begin{bmatrix}
p_1^*
p_2^*
\end{bmatrix}.
\tag{15}
\]

While we do not have boundary conditions for \( p^*(t) \) we have from (11)

\[
1 + x_2^*(t)p_1^*(t)(1 - 2\epsilon \bar{u}) - x_1^*(t)p_2^*(t)(1 + \epsilon \bar{u}) = 0.
\tag{16}
\]

The primary difficulty faced in the application of the Pontryagin minimum principle is that the necessary conditions result in a two-point-boundary-value-problem. While these problems are typically difficult to solve analytically, here, after inspecting (15) and (16), we obtain the
following equation for $p^*(t)$ that satisfies both (15) and (16)

\[
\begin{align*}
  p_1^*(t) &= \frac{1}{\bar{x}_1 \omega} \cos(\omega t + \phi), \\
  p_2^*(t) &= -\frac{1}{\bar{x}_1 (1 + \epsilon \bar{u})} \sin(\omega t + \phi).
\end{align*}
\]  

(17)

Solution to the time-optimal control problem

We now need to determine the condition on $\phi$ so that equations for the state and costate vectors satisfy all the necessary conditions for optimality. Using the transversality condition (12), since the target set is a circle, at the final time the state and costate vector are parallel to each other. Therefore

\[ p^*(T^*) = \lambda x^*(T^*). \]

We then have,

\[ \frac{p_1^*(T)}{x_1^*(T)} = \frac{p_2^*(T)}{x_2^*(T)}. \]

(18)

Substituting from (14) and (17) in (18) and using the assumption made earlier that $\epsilon \ll 1$, we obtain that $\sin(\phi) = 0$. Therefore either $\phi = 2n\pi$ or $\phi = (2n + 1)\pi$, where $n = 0, 1, 2 \ldots$.

The choice for $\phi$ between these two possibilities is made by determining if the radius of the target circle is greater than the initial angle $\bar{x}_1$, in which case the swing has to be pumped to increase its amplitude of motion, or if the radius of the target circle is smaller than the initial angle, in which case the amplitude of oscillation of the swing has to be reduced.

For the case where the amplitude of oscillation of the swing is to be increased, substituting
\( \phi = (2n + 1)\pi \) in (17) and using (13) we have

\[
\begin{align*}
u^*(t) & = \text{sgn}[2p_1(t)x_2(t) + p_2(t)x_1(t)] \\
& = \text{sgn}[\frac{2}{1 - 2\epsilon\Delta} \sin(\omega t) \cos(\omega t) + \frac{1}{1 + \epsilon\Delta} \sin(\omega t) \cos(\omega t)] \\
& = -\text{sgn}[x_1^*(t)x_2^*(t)].
\end{align*}
\]

Similarly for the case where the amplitude of oscillations of the swing is to be decreased, substituting \( \phi = 2n\pi \) in (17) we obtain

\[
u^*(t) = \text{sgn}[x_1^*(t)x_2^*(t)].
\]

The optimal trajectory for the two cases is shown in Figure 3. We have finally obtained after the application of the minimum principle the time-optimal control law, which states that the controller switches between +1 and -1 when the state crosses either coordinate axis. The optimal control strategy is thus in feedback form and is identical to the one outlined in the introduction.

**Optimal Control: Nonlinear Case**

The solution to the optimal control problem for the nonlinear system (5) is obtained using techniques from the field of geometric optimal control. A detailed discussion of this area is not attempted here and the interested reader is guided to [29]. Here we mainly present a rough sketch of the methods used to obtain a solution to our optimal control problem. Let us then turn back to (5), which we restate for the sake of convenience,

\[
\begin{align*}
\dot{x}_1 & = \frac{x_2}{l^2}, \\
\dot{x}_2 & = -gl \sin(x_1).
\end{align*}
\]
The system equation with the length $l$ of the pendulum as the control input is

$$
\dot{x} = h(x, u) = \begin{pmatrix}
\frac{x_1^2}{u^2} \\
-gu \sin(x_1)
\end{pmatrix},
$$

(19)

where the control $u$ takes values on some interval $[l_-, l_+]$ with $0 < l_- < l_+$, corresponding to the fact that the center of mass cannot coincide with the point of rotation ($l_+ > 0$), and that there is a bound on the lower position of the rider ($l_+ < +\infty$).

The main difficulty in analyzing (19) is the nonlinearity in the control input. However defining, $h^\pm(x) = h(x, l_\pm)$, we observe that the set of admissible velocities at a point $x$ is contained in the triangle obtained as a convex hull of the three vectors $0$, $h^-(x)$, and $h^+(x)$, as shown in Figure 4. Therefore every admissible velocity at $x$ for (19) is realized with a greater magnitude by a point on the segment joining $h^-(x)$ and $h^+(x)$, implying that points on this segment result in faster trajectories [30]. This observation enables us to recast the original optimal control problem for the nonlinear system (19) by considering an auxiliary system with velocities in the segment joining $h^-(x)$ and $h^+(x)$. To illustrate this idea we introduce the auxiliary system

$$
\dot{x} = F(x) + G(x)u, \quad |u| \leq 1,
$$

(20)

where

$$
F(x) = \begin{pmatrix}
d x_2 x_1 \\
-\frac{b}{2} b \sin(x_1)
\end{pmatrix}, \quad G(x) = \begin{pmatrix}
b c x_1 \\
\frac{c}{2} c \sin(x_1)
\end{pmatrix},
$$

and $a = (l_+ l_-)^2$, $b = l_+ + l_-$, $c = l_- - l_+$, and $d = l_+^2 + l_-^2$. The auxiliary system corresponds exactly to having velocities in the segment joining $h^-(x)$ with $h^+(x)$, which is easily checked by verifying that $F(x) + G(x) = h^+(x)$ and $F(x) - G(x) = h^-(x)$.

Dimension-two systems of type (20) have been studied extensively in [29] and [31–33]. For these systems a detailed analysis of the structure of optimal trajectories and a synthesis of the
optimal controls is possible. A general method to synthesize optimal controls on a 2-D manifold is illustrated in [31], along with a classification of various singularities that may appear in optimal flows.

A key role in the analysis of optimal trajectories for (20) is played by the two functions

\[
\Delta_A(x) = \det(F(x), G(x)) = F_1(x)G_2(x) - F_2(x)G_1(x),
\]

\[
\Delta_B(x) = \det(G(x), [F, G](x)) = G_1(x)[F, G]_2(x) - G_2(x)[F, G]_1(x),
\]

where \([F, G]\) denotes the Lie bracket of \(F\) and \(G\) and is given by, \([F, G] = \nabla G \cdot F - \nabla F \cdot G\). In particular if both functions do not vanish in an open set \(\Omega\) of the phase-space then every optimal trajectory in \(\Omega\) is bang-bang with at most one switching [29]. This situation holds on each quadrant for the linearized system treated in the previous section. For (20) we compute

\[
\Delta_A(x) = \frac{gc(d + b^2)}{4a}x_2 \sin(x_1),
\]

\[
\Delta_B(x) = \frac{gc^2(d + b^2)}{8a} \left( \frac{b}{a}x_2 \cos(x_1) + g \sin^2(x_1) \right),
\]

implying only bang-bang controls for \(x_1 \in [-\pi/2, \pi/2]\) and at most one switching for every quadrant.

The link between systems (19) and (20) is given by the following theorem [30].

**Theorem 1** Consider two points \(x_{ini}\) and \(x_{fin}\), denoting the prescribed initial and final state.

1. If there exists a bang-bang time-optimal control \(u\) for (20) steering \(x_{ini}\) to \(x_{fin}\), then there exists a time-optimal control \(v\) for (19) corresponding to the same trajectory \(\gamma\) of \(u\), that is \(h(\gamma(t), v(t)) = F(\gamma(t)) + G(\gamma(t))u(t)\).

2. If the time-optimal control \(u\) for (20) is not bang-bang then the time-optimal control for (19) does not exist. But if \(T\) denotes the time taken by \(u\), then for each \(\varepsilon > 0\) there exists
a control $v$ steering $x_{ini}$ to $x_{fin}$ in time $T + \varepsilon$, such that $v(t) \in \{l_{\pm}\}$ for every $t$.

Roughly speaking the theorem states that either a bang-bang optimal control exists for (20) and that the same trajectories are optimal for (19), or the optimal control for (20) is not bang-bang, and then the optimal control for (19) does not exist but that the minimum time trajectory is approximated by bang-bang trajectories. Therefore to solve optimal control problems for (19) it is sufficient to construct optimal controls for (20). The trajectories minimizing and maximizing the final oscillation correspond to bang-bang controls, with the control input switching each time the state crosses either coordinate axis, exactly as indicated by the solution to the linearized problem discussed earlier.

As an example Figure 5 shows the optimal synthesis starting from a point on the positive $x_1$ axis. This figure represents time-optimal trajectories starting from $x_{ini}$ and is constructed using the methods developed in [31]. There are two key curves, characterizing the singularities of the synthesis, that start from the initial point and correspond to constant control $\pm 1$ up to the first intersection with the $x_2$ axis. These two trajectories are the so-called abnormal extremals [31], and change control each time they cross the set where the function $\Delta_A$ vanishes. The dashed green lines indicate the switching curves where the control input changes sign along the optimal trajectories.

**Conclusion**

Children playing on a swing all follow a similar strategy, they stand up at the lowest point and squat at the highest. In the limit that standing and squatting can be carried out instantaneously, we have shown, using the Pontryagin minimum principle and techniques from the field of geometric optimal control, that this pumping strategy is time optimal for the problem of maximizing the amplitude of oscillation of a swing. Given the exuberant nature of youth,
it is reasonable to believe that children on a swing try to go as high as possible as quickly as possible and that the cost function that is being minimized is indeed time. Thus one of the key concepts in biomechanics, that humans while performing skilled tasks act as self-optimizing machines, is shown to hold in the case of pumping of a swing.

Acknowledgements

We would like to thank Prof. Richard Rand of Cornell University for many interesting discussions regarding the problem of pumping of a swing.
References


Figure 1: Simulation of Mathieu’s equation for $\delta = 1/4$, $\epsilon = 0.1$, and $L = 10$. The solution to Mathieu’s equation is unbounded for certain values of the parameters. The figure is for the case when the frequency of variation of the length of the pendulum is twice the pendulum’s natural frequency.
Figure 2: Pumping strategy to increase the amplitude. Children playing on a swing typically follow this pumping strategy, standing up at the lowest point and squatting at the highest point. Interchanging the standing and squatting maneuvers results in a decrease in the amplitude of oscillation.
Figure 3: Time-optimal trajectories for linearized system. Left and right plot show time-optimal trajectories to reach the target circle with a radius smaller and greater than the initial angle respectively. The coordinate axes are the switching curves, where the control input switches between +1 and -1. The squatting position corresponds to $u = +1$ and the standing position corresponds to $u = -1$. 
Figure 4: Set of admissible velocities. The set of admissible velocities for state $x$ is contained in the triangle obtained as a convex hull of the three vectors $0$, $h^-(x)$, and $h^+(x)$. The figure is constructed for the case where $x_1 < 0$ and $x_2 > 0$. 
Figure 5: Time-optimal trajectories for the nonlinear system (19). The initial point is on the positive $x_1$ axis. Note that trajectories minimizing (red) and maximizing (blue) the amplitude of oscillation switch control input when the state crosses either coordinate axis. The dashed green lines indicate the switching curves.
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