Particular solutions
for a class of ODE related
to the L-exponential functions

by Gabriella Bretti(*) and Pierpaolo Natalini (†)

(*) Università di Roma “La Sapienza”,
Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Via A. Scarpa, 14 - 00161 Roma, Italia

(†) Università degli Studi Roma Tre, Dipartimento di Matematica,
Largo San Leonardo Murialdo, 1, 00146 - Roma, Italia

Abstract – Particular solutions of a class of higher order ordinary differential equations, with non-constant coefficients, are determined by exploiting the properties of the Laguerre exponentials functions introduced by G. Dattoli and P.E. Ricci in [4].

AMS CLASSIFICATION: 33B10, 33EXX, 34A05, 34A30.
KEY WORDS: L-exponentials, Ordinary linear differential equations.
1 Introduction

In a recent article G. Dattoli and P.E. Ricci [4] introduced a class of functions, called Laguerre exponentials (shortly L-exponentials) which satisfy, with respect to the Laguerre derivative operator \( D_L := DxD \), and its generalizations, the same eigenvalue property as the ordinary exponential does with respect to the ordinary derivative operator \( D \).

In the same paper, the L-circular and L-hyperbolic functions are also defined, in such a way that, defining in the space of analytic functions a differential isomorphism by means of the correspondence:

\[
D \rightarrow D_L = DxD, \quad x \rightarrow D_x^{-1},
\]

(1.1)

where

\[
D_x^{-1} f(x) := \int_0^x f(t) dt
\]

(1.2)

denotes the anti-derivative so that

\[
D_x^{-n}(1) = \frac{x^n}{n!},
\]

(1.3)

the above functions correspond to the ordinary circular and hyperbolic ones.

As an example, according to such an isomorphism, the exponential function \( e^x \) corresponds to the Tricomi function

\[
C_0(-x) := \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2},
\]

the powers \((x + y)^n\), (which are usually interpreted as the Hermite polynomials \( H_{n}^{(1)}(x, y) \) of order one, see e.g. [2], [3], [5], [6]), correspond to the two variable (homogeneous) Laguerre polynomials

\[
\mathcal{L}_n(-x, y) := n! \sum_{r=0}^{n} \frac{y^{n-r}x^r}{(n-r)!(r!)^2}
\]

and so on.

The iteration of such an isomorphism permits to introduce further maps in the space of analytic functions.
In this article, exploiting the conservation rule determined by the above mentioned isomorphism, we show how to construct in a simple way particular solutions to a class of ordinary differential equations with non constant coefficients.

2 Recalling the L-exponentials

For every positive integer $n$, the relative L-exponential function is defined as

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}.$$  

This function reduces to the classical exponential function when $n = 0$, so that we can put by definition $e_0(x) := e^x$. For the function $e_1(x)$ we have the relation $e_1(ax) := C_0(-ax)$.

Considering the differential operator

$$D_{nL} := Dx \cdots Dx Dx = D \left( xD + x^2D^2 + \ldots x^nD^n \right)$$  

(2.1)

$$= S(n+1,1)D + S(n+1,2)x D^2 + \ldots + S(n+1,n+1)x^n D^{n+1},$$

where $S(n+1,1), S(n+1,2), \ldots, S(n+1,n+1)$ are the Stirling numbers of second kind (see [1], page 835 for an extended table), it was proved (see [4]) the following theorem

**Theorem 2.1** Let $a$ be a real or complex arbitrary constant. Then the function $e_n(ax)$ is an eigenfunction of the operator $D_{nL}$ i.e.

$$D_{nL} e_n(ax) = ae_n(ax).$$  

(2.2)

For $n = 0$ we have $D_{0L} := D$ and therefore the (2.2) becomes the classical relation of exponential

$$D e_0(ax) = ae_0(ax).$$

For $n = 1$ and $n = 2$ the operators $D_L$ and $D_{2L}$ are respectively

$$D_L := Dx D = D + xD^2$$  

(2.3)

and

$$D_{2L} := Dx Dx D = D \left( xD + x^2D^2 \right) = D + 3xD^2 + x^2D^3.$$  

(2.4)
3 A preliminary result

In this section we will generalize the known property of the Laguerrian derivative $D x D$, according to which we can write:

$$ (D x D)^{h} = D^{h} x^{h} D^{h}, \quad \forall \text{ integer } h. \quad (3.1) $$

Let us note that a similar result holds true for the operator $D x D x D$, since its application to every power yields:

$$ D x D x D x^{k} = k^{3} x^{k-1}, $$

so that

$$ (D x D x D)^{h} x^{k} = k^{3}(k - 1)^{3} \cdots (k - h + 1)^{3} x^{k-h}, $$

and the same result can be found by applying $D^{h} x^{h} D^{h} x^{h} D^{h}$, since

$$ D^{h} x^{h} D^{h} x^{h} D^{h} x^{k} = D^{h} x^{h} D^{h} x^{h} D^{h} x^{h} k(k - 1) \cdots (k - h + 1)x^{k-h} $$

$$ = \cdots = k^{3}(k - 1)^{3} \cdots (k - h + 1)^{3} x^{k-h}. $$

The same technique can be applied in order to state the following general result.

**Theorem 3.1** Consider the operator

$$ D x D \cdots D x D, \quad \text{(containing } n \text{--times } x, \text{ and } (n + 1)\text{--times } D). $$

Then, for every integer $h$, the following identity holds true:

$$ (D x D \cdots D x D)^{h} = D^{h} x^{h} D^{h} x^{h} \cdots D^{h} x^{h} D^{h}. \quad (3.2) $$

**Proof.** By using linearity, it is sufficient to prove the result for powers.

It is easily seen that

$$ (D x D \cdots D x D)^{h} x^{k} = k^{n+1}(k - 1)^{n+1} \cdots (k - h + 1)^{n+1} x^{k-h}, $$

and, by induction:

$$ D^{h} x^{h} D^{h} x^{h} \cdots D^{h} x^{h} D^{h} x^{k} = D^{h} x^{h} D^{h} x^{h} \cdots D^{h} x^{h} k(k - 1) \cdots (k - h + 1)x^{k-h} $$

$$ = D^{h} x^{h} D^{h} x^{h} \cdots D^{h} x^{h} k^{2}(k - 1)^{2} \cdots (k - h + 1)^{2} x^{k-h} \quad (3.3) $$

$$ = \cdots = k^{n+1}(k - 1)^{n+1} \cdots (k - h + 1)^{n+1} x^{k-h}. $$

4
4 Solutions for higher order special differential equations

Consider first the differential operator $D_L$. It is easily proved, by induction, that

$$D_L^h = D^h x^h D^h = \sum_{r=0}^{h} \binom{h}{r} D^{h-r} x^h D^{h+r}$$

$$= \sum_{r=0}^{h} \binom{h}{r} \frac{h!}{r!} x^r D^{h+r}$$

$$= (h!)^2 \sum_{r=0}^{h} \frac{x^r}{(r!)^2(h-r)!} D^{h+r}. \quad (4.1)$$

Then, we can state the following theorem:

**Theorem 4.1** Consider the differential equation, of order $2N$:

$$\sum_{h=0}^{N} a_h D_L^h \ y = 0, \quad (4.2)$$

where the coefficients $a_h$ are (real or complex) constants, and the characteristic equation

$$\sum_{h=0}^{N} a_h \lambda^h = 0. \quad (4.3)$$

Then, denoting by $\lambda_1, \lambda_2, \ldots, \lambda_N$ the corresponding roots of the above equation, for every choice of the arbitrary constants $c_1, c_2, \ldots, c_N$ the differential equation admits the solution:

$$y(x, c_1, c_2, \ldots, c_N) = c_1 e_1(\lambda_1 x) + c_2 e_1(\lambda_2 x) + \ldots + c_N e_1(\lambda_N x). \quad (4.4)$$

Obviously, the dimension of the corresponding linear space of solutions is at most $N$, and the maximal dimension is reached if and only if all the roots of the characteristic equation are distinct.

Note that, according to the above eq. (4.1), eq. (4.2) becomes:

$$\sum_{h=0}^{N} a_h \sum_{r=0}^{h} \frac{(h!)^2}{((r!)^2(h-r)!)} x^r D^{r+h} y$$
\[= \sum_{h=0}^{N} a_h \sum_{r=0}^{2h} \frac{(h!)^2}{((r-h)!)^2(2h-r)!} x^{r-h} D^r y \]  

\[= 2N \left( \sum_{r=0}^{\min\{r,N\}} \sum_{h=\left\lceil \frac{r+1}{2} \right\rceil}^{h} a_h \frac{(h!)^2}{((r-h)!)^2(2h-r)!} x^{r-h} \right) D^r y = 0, \]

which is a linear differential equation of order \(2N\), with non-constant coefficients.

**Proof.** Substituting the expression (4.4) of \(y\) into the left-hand side of (4.2) we have

\[
\sum_{h=0}^{N} a_h D^h \left[ c_1 e_1(\lambda_1 x) + c_2 e_1(\lambda_2 x) + \ldots + c_N e_1(\lambda_N x) \right] \\
= \sum_{h=0}^{N} a_h \left[ c_1 \lambda_1^h e_1(\lambda_1 x) + c_2 \lambda_2^h e_1(\lambda_2 x) + \ldots + c_N \lambda_N^h e_1(\lambda_N x) \right] \\
= \sum_{k=1}^{N} c_k e_1(\lambda_k x) \sum_{h=0}^{N} a_h \lambda_k^h = 0.
\]

It is easy to see that, if \(\lambda_i \neq \lambda_j\), the functions \(e_1(\lambda_i x)\) and \(e_1(\lambda_j x)\) are linearly independent. In fact, for every pair of real (or complex) constants \(c_1, c_2\) we have

\[c_1 e_1(\lambda_i x) + c_2 e_1(\lambda_j x) = \sum_{k=0}^{\infty} \left( c_1 \lambda_i^k + c_2 \lambda_j^k \right) \frac{x^k}{(k!)^2} = 0\]

if and only if \(c_1 \lambda_i^k + c_2 \lambda_j^k = 0\) for every \(k = 0, 1, \ldots\), from which we obtain that \(c_1 = c_2 = 0\).

The above results can be extended to the general differential operator \(D_{nL}\). First we consider the operator \(D_{2L}\). Then, for every positive integer \(h\), we have:

\[D_{2L}^h = \sum_{r=0}^{h} \frac{(h!)^2}{(r!)^2(2h-r)!} D^{h-r} x^r D^h \]

\[= \sum_{r=0}^{h} \frac{(h!)^2}{(r!)^2(2h-r)!} x^r \sum_{s=0}^{h+r} \frac{(h + r)!}{s! (h + r - s)! (s - r)!} x^{s-r} D^{h+s} \]

\[= (h!)^2 \sum_{r=0}^{h} \frac{x^r}{(r!)^2(2h-r)!} \sum_{s=0}^{h+r} \frac{(h + r)!}{s! (h + r - s)! (s - r)!} x^{s-r} D^{h+s} \]

\[= (h!)^3 \sum_{r=0}^{h} \frac{(h + r)!}{(r!)^2(2h-r)!} \sum_{s=0}^{h+r} \frac{x^s}{s! (h + r - s)! (s - r)!} D^{h+s}. \]
The formula (4.6) can be extended to the general case. In fact, the following result holds true:

**Theorem 4.2** For every pair of positive integers \( n \) \((n \geq 2)\) and \( h \), the following formula holds true:

\[
D_h^{n} = (h!)^{n+1} \sum_{k_1=0}^{h} \frac{(h+k_1)!}{(k_1)!^2(h-k_1)!} \sum_{k_2=k_1}^{h+k_1} \frac{(h+k_2)!}{k_2!(h+k_1-k_2)!(k_2-k_1)!} \cdot \sum_{k_{n-1}=k_{n-2}}^{h+k_{n-1}} \frac{(h+k_{n-1})!}{k_{n-1}!(h+k_{n-2}-k_{n-1})!(k_{n-1}-k_{n-2})!} \cdot \sum_{k_n=k_{n-1}}^{h+k_n} \frac{x^{k_n}}{k_n!(h+k_{n-1}-k_n)!(k_n-k_{n-1})!} D^{h+k_n}.
\]  

(4.7)

**Proof.** We proceed the proof by induction. For \( n = 2 \) (4.7) becomes (4.6). Now we suppose (4.7) true and prove the same one for \( n + 1 \) instead of \( n \). Then we have

\[
D_{n+1}^{h} = D_n^{h} x^h D^h
\]  

(4.8)

Since

\[
D^{h+k_n} x^h D^h = \sum_{k_{n+1}=0}^{h+k_n} \binom{h+k_n}{k_{n+1}} D^{h+k_n-k_{n+1}} x^h D^{h+k_{n+1}}
\]  

(4.9)

substituting (4.9) into (4.8) we have the proof.

Now we can state the following theorem, the proof of which is obtained following the same methods of Theorem 4.1.

**Theorem 4.3** Consider the differential equation, of order \((n+1)N\):

\[
\sum_{h=0}^{N} a_h D_n^h y = 0,
\]  

(4.10)
where the coefficients $a_h$ are (real or complex) constants, and the characteristic equation

$$\sum_{h=0}^{N} a_h \lambda^h = 0. \quad (4.11)$$

Then, denoting by $\lambda_1, \lambda_2, \ldots, \lambda_N$ the corresponding roots of the above equation, for every choice of the arbitrary constants $c_1, c_2, \ldots, c_N$ the differential equation admits the solution:

$$y(x, c_1, c_2, \ldots, c_N) = c_1 e_n(\lambda_1 x) + c_2 e_n(\lambda_2 x) + \ldots + c_N e_n(\lambda_N x). \quad (4.12)$$

Proof. Note that, according to the above eq. (4.7), eq. (4.10) becomes

$$\sum_{h=0}^{N} a_h \lambda^h = 0 \quad (4.11)$$

$$\sum_{h=0}^{N} \sum_{k_1=0}^{h} \frac{(h+k_1)!}{(k_1)!^2} \frac{(h+k_2)!}{k_1!^2(k_2!)} \frac{(h+k_3)!}{k_3!(h+k_2-k_3)!(k_3-k_2)!} \ldots \frac{(h+k_{n-2})!}{k_{n-2}!(h+k_{n-3}-k_{n-2})!(k_{n-3}-k_{n-2})!} \frac{(h+k_{n-1})!}{k_{n-1}!(h+k_{n-2}-k_{n-1})!(k_{n-2}-k_{n-2})!} \frac{x^{k_n}}{k_n!(h+k_{n-1}-k_n)!(k_n-k_{n-1})!} D^{h+k_n} y = 0. \quad (4.13)$$

The last sum on the left hand side of (4.13) can be rewritten in the following way

$$\sum_{k_n=k_{n-1}+h}^{2h+k_{n-1}} \frac{x^{k_n}}{k_n!(2h+k_{n-1}-k_n)!(k_n-h-k_{n-1})!} D^{k_n} y.$$ 

Then by interchanging the order of the two last sums on the left hand side the equation (4.13) becomes

$$\sum_{h=0}^{N} \sum_{k_1=0}^{h} \frac{(h+k_1)!}{(k_1)!^2} \frac{(h+k_2)!}{k_1!^2(k_2!)} \frac{(h+k_3)!}{k_3!(h+k_2-k_3)!(k_3-k_2)!} \ldots \frac{(h+k_{n-2})!}{k_{n-2}!(h+k_{n-3}-k_{n-2})!(k_{n-3}-k_{n-2})!} \frac{(h+k_{n-1})!}{k_{n-1}!(h+k_{n-2}-k_{n-1})!(k_{n-2}-k_{n-2})!} \frac{x^{k_n}}{k_n!(h+k_{n-1}-k_n)!(k_n-k_{n-1})!} D^{h+k_n} y = 0.$$
\[
\begin{align*}
\cdots \sum_{k_{n-2}=k_{n-3}}^{h+k_{n-3}} \frac{(h+k_{n-2})!}{k_{n-2}!(h+k_{n-3} - k_{n-2})!(k_{n-2} - k_{n-3})!} \\
\cdot \sum_{k_{n}=k_{n-2}+h}^{3h+k_{n-2}} \min\{k_{n-1},h+k_{n-2}\} \sum_{k_{n-1}=\max(k_{n-2},k_{n}-2h)}^{\min\{k_{n-1},h+k_{n-2}\}} \frac{(h+k_{n-1})!}{k_{n-1}!(h+k_{n-2} - k_{n-1})!(k_{n-1} - k_{n-2})!} \\
\cdot \sum_{k_{n-1}=\max(k_{n-2},k_{n}-2h)}^{\min\{k_{n-1},h+k_{n-2}\}} \frac{x^{k_{n}-h}}{(k_{n} - h)!(2h+k_{n-1} - k_{n})!(k_{n} - h - k_{n-1})!} D^{k_{n}} y = 0.
\end{align*}
\]

Again, by interchanging the order of the two sums \(\sum_{k_{n-2}=k_{n-3}}^{h+k_{n-3}}\) and \(\sum_{k_{n}=k_{n-2}+h}^{3h+k_{n-2}}\) on the left hand side the equation (4.14) becomes

\[
\begin{align*}
\sum_{h=0}^{N} a_{h} \frac{(h!)^{n+1}}{(k_{1}!)^{2}(h-k_{1})!} \sum_{k_{1}=0}^{h+k_{2}} \frac{(h+k_{1})!}{k_{2}!(h+k_{1} - k_{2})!(k_{2} - k_{1})!} \\
\cdot \sum_{k_{3}=k_{2}}^{h+k_{2}} \frac{(h+k_{3})!}{k_{3}!(h+k_{2} - k_{3})!(k_{3} - k_{2})!} \\
\cdot \sum_{k_{n}=k_{n-3}+h}^{4h+k_{n-3}} \min\{k_{n-1},h+k_{n-3}\} \sum_{k_{n-1}=\max(k_{n-2},k_{n}-2h)}^{\min\{k_{n-1},h+k_{n-2}\}} \frac{(h+k_{n-2})!}{k_{n-2}!(h+k_{n-3} - k_{n-2})!(k_{n-2} - k_{n-3})!} \\
\cdot \sum_{k_{n-1}=\max(k_{n-2},k_{n}-2h)}^{\min\{k_{n-1},h+k_{n-2}\}} \frac{(h+k_{n-1})!}{k_{n-1}!(h+k_{n-2} - k_{n-1})!(k_{n-1} - k_{n-2})!} \\
\cdot \sum_{k_{n-1}=\max(k_{n-2},k_{n}-2h)}^{\min\{k_{n-1},h+k_{n-2}\}} \frac{x^{k_{n}-h}}{(k_{n} - h)!(2h+k_{n-1} - k_{n})!(k_{n} - h - k_{n-1})!} D^{k_{n}} y = 0.
\end{align*}
\]

Therefore, by proceeding in this way until to the first sum on the left hand side of (4.15), we obtain the following final expression for the differential equation

\[
\sum_{k_{n}=0}^{(n+1)N} A_{h,k_{1},...,k_{n}}(x) D^{k_{n}} y = 0,
\]

where

\[
A_{h,k_{1},...,k_{n}}(x) = \sum_{h=\left\lfloor \frac{k_{n}}{n+1} \right\rfloor}^{\min\{k_{n},N\}} \sum_{k_{1}=\max\{0,k_{n}-nh\}}^{\min\{k_{n},h\}} \sum_{k_{2}=\max\{k_{1},k_{n}-(n-1)h\}}^{\min\{k_{n},h+k_{1}\}} \cdots \sum_{k_{n-1}=\max\{k_{n-2},k_{n}-2h\}}^{\min\{k_{n-1},h+k_{n-2}\}} \frac{(h+k_{1})! \cdots (h+k_{n-1})!}{(k_{1}!)^{2} k_{2}! \cdots k_{n-1}! (k_{2} - k_{1})! \cdots (k_{n-1} - k_{n-2})!} \\
\cdot \sum_{k_{n-1}=\max\{k_{n-2},k_{n}-2h\}}^{\min\{k_{n-1},h+k_{n-2}\}} \frac{1}{(h+k_{1}-k_{2})! \cdots (h+k_{n-2} - k_{n-1})!} \\
\cdot \frac{1}{(h-k_{1})! (k_{n}-h)!(2h+k_{n-1} - k_{n})!(k_{n} - h - k_{n-1})!} x^{k_{n}-h}.
\]
Eq. (4.16) is a linear differential equation of order \((n + 1)N\), with non-constant coefficients. For \(n = 2\) eq. (4.16) and eq. (4.17) become respectively
\[
\sum_{s=0}^{3N} A_{h,r,s}(x) D^s y = 0,
\]
\[
A_{h,r,s}(x) = \sum_{h=\left\lfloor \frac{r}{2} \right\rfloor}^{\min\{s,N\}} \sum_{r=\max\{0,s-2h\}}^{\min\{s-h,h\}} a_h \frac{(h!)^3(h+r)!}{(r!)^2(h-r)!(s-h)!(2h+r-s)!(s-h-r)!} x^{s-h}.
\]
This completes the proof of the Theorem 4.3.

**Acknowledgments.** – It is our pleasure to recognize the help of Prof. Paolo E. Ricci for introducing us to the knowledge of the L-exponential functions.

**References**


10
Address:

Gabriella BRETTI
Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Università degli Studi di Roma "La Sapienza"
Via A. Scarpa, 14 – 00161 Roma (Italia)
e-mail: bretti@dmmm.uniroma1.it

Pierpaolo NATALINI
Dipartimento di Matematica – Università degli Studi Roma Tre
Largo San Leonardo Murialdo, 1 – 00146 - ROMA
e-mail: natalini@mat.uniroma3.it