A numerical scheme for a hyperbolic relaxation model on networks

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Abstract. In this article, we consider a simple relaxation hyperbolic system on networks which models the movement of fibroblasts on an artificial scaffold. After proving the uniqueness of stationary solutions with a given total mass, we present an adapted numerical scheme which takes care of boundary conditions and display some numerical tests.

Keywords: relaxation hyperbolic system, network, finite differences, boundary conditions, stationary solutions, fibroblasts, scaffold

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1. INTRODUCTION

In this article, we consider a relaxation system, that is to say a simplified version of the hyperbolic system for chemotaxis studied in [3], on a network. The aim is to find an accurate numerical scheme to solve this hyperbolic system set on each arc of the network as a first step for the study of the complete chemotaxis system with source. Indeed, as shown in [4] in the case of a single interval, to find a numerical scheme for the complete hyperbolic system and especially well-adapted boundary conditions is a delicate task.

Therefore, let us define a network or a graph \( G = (\mathcal{N}, \mathcal{A}) \), which is composed by two finite sets, a set of nodes (or vertices) \( \mathcal{N} \) and a set of arcs (or edges) \( \mathcal{A} \) such that an arc connects a pair of nodes. Our graph is non-oriented, that is to say arcs can be considered in both directions, but we need to fix an artificial orientation in order to give a sign to velocity. On each arc of the graph, we set a scalar system of two transport equations, coupled by their source terms. These equations model respectively the evolution of the density of cells going in one direction. Systems on two arcs connected at a node are coupled through transmission conditions at the node.

This system can be viewed as a simple model for the movement of fibroblasts on an artificial scaffold, see also [1]. Fibroblasts are stem cells that play a crucial role in wound healing by synthesizing extracellular matrix and collagen and migrating toward the wound via chemotaxis. In tissue engineering, researchers aim at developing artificial scaffold made of collagen to mimic the extracellular matrix. This process enables to accelerate wound healing and minimize scarring [5]. The arcs of the network stand for the fibers of the scaffold and the transport equations give the evolution of the density of fibroblasts on each fiber.

In Section 2, we will describe in details the equations and the relative boundary and transmission conditions. Then in the next section, we prove in a simple case that non-null stationary solutions are unique up to a constant which is given by the initial mass of our solution. Finally, we propose in Section 4 a relevant numerical scheme and we display numerical tests in Section 5.

2. A SIMPLE HYPERBOLIC RELAXATION SYSTEM ON A NETWORK

Let us consider an arc \( a_i \in \mathcal{A} \) of length \( L_i \) of the graph and the following hyperbolic system where \( u^+_i \) denote the density of cells following the orientation of the arc \( a_i \) (resp. the density of cells going in the opposite direction) :

\[
\begin{align*}
\frac{du^+_i}{dx} + \lambda_i u^+_i(x,t) &= \frac{1}{2\lambda_i} \left( (\alpha_i - \lambda_i) u^+_i(x,t) + (\alpha_i + \lambda_i) u^-_i(x,t) \right), \\
\frac{du^-_i}{dx} - \lambda_i u^-_i(x,t) &= -\frac{1}{2\lambda_i} \left( (\alpha_i - \lambda_i) u^+_i(x,t) + (\alpha_i + \lambda_i) u^-_i(x,t) \right),
\end{align*}
\]

for \( x \in [0, L_i], t > 0 \) (1)
where \( T_i^\pm = \frac{1}{2\lambda_i}(\alpha_i \mp \lambda_i) \) are the turning rates (namely the probability of cells to change direction). Let us notice that we assume that all cells on the arc \( a_i \) have the same velocity \( \lambda_i \).

We also rewrite the previous system (1) using two other variables, namely the total density of cells \( u_i \) and the total flux \( v_i \) defined by \( u_i = u_i^+ + u_i^- , v_i = \lambda_i(u_i^+ - u_i^-) \), which yields

\[
\begin{cases}
  u_i(t,x,t) + v_i(x,t) = 0, \\
  v_i(t,x,t) + \lambda_i^2 u_i(x,t) = \alpha_i u_i(x,t) - v_i(x,t), \quad x \in [0,L_i], \quad t > 0.
\end{cases}
\]  

(2)

For analytical reasons, we assume also that for all indices \( i \) such that \( a_i \in \mathcal{A} \), \( \lambda_i > |\alpha_i| \).

Now, let us deal with boundary and transmission conditions. We first set the boundary conditions at an external node \( n_k \in \mathcal{N} \) for the arc \( a_i \in \mathcal{A} \) which begins or ends at \( n_k \) as in the case of chemotaxis on a single interval [3], namely with the following no-flux boundary conditions:

\[
u_i^+(.,t) = u_i^-(.,t) \quad \text{(which is equivalent to } v_i(.,t) = 0) .
\]  

(3)

Let us now consider an internal node \( n_k \in \mathcal{N} \). According to the orientation we put on each arc, there are two different types of arcs beginning or ending at node \( n_k \) : the incoming arcs, whose set is denoted by \( \mathcal{I}_k \) and outgoing arcs, whose set is denoted by \( \mathcal{O}_k \). Therefore the transmission conditions will be the following at a node \( n_k \) for the arc \( a_i \) which begins or ends at \( n_k \):

\[
\begin{cases}
  u_i^+(L_i,t) = \sum_{j \in \mathcal{I}_k} \xi_{i,j} u_j^+(L_j,t) + \sum_{j \in \mathcal{O}_k} \xi_{i,j} u_j^-(0,t), \quad \text{if } i \in \mathcal{I}_k \\
  u_i^+(0,t) = \sum_{j \in \mathcal{I}_k} \xi_{i,j} u_j^+(L_j,t) + \sum_{j \in \mathcal{O}_k} \xi_{i,j} u_j^-(0,t), \quad \text{if } i \in \mathcal{O}_k,
\end{cases}
\]  

(4)

where \( \xi_{i,j} \in [0,1] \). Let us remark that for an incoming arc \( a_i, L_i \) is the abcissa of the node, whereas it is 0 for an outgoing arc. These transmission conditions mean that a cell that arrives at a node can choose between the different directions with given probabilities \( \xi_{i,j} \). We impose the following condition, where \( \lambda_i \) is the velocity of arc \( a_i \) :

\[
\sum_{j \in \mathcal{I}_k \cup \mathcal{O}_k} \xi_{i,j} \lambda_j = \lambda_i
\]  

(5)

which gives at a node \( n_k \) the flux conservation

\[
\sum_{i \in \mathcal{I}_k} \lambda_i(u_i^+(L_i,t) - u_i^-(L_i,t)) = \sum_{i \in \mathcal{O}_k} \lambda_i(u_i^+(0,t) - u_i^-(0,t)) \quad \text{(or in the } v \text{ variable } \sum_{i \in \mathcal{I}_k} v_i(L_i,t) = \sum_{i \in \mathcal{O}_k} v_i(0,t) ).
\]  

(6)

This leads to the following mass conservation for the whole network :

\[
\sum_{a_i \in \mathcal{A}} \int_0^{L_i} u_i(x,t)dx = \sum_{a_i \in \mathcal{A}} \int_0^{L_i} u_i(x,0)dx = \mu_0, \quad \text{for all } t > 0.
\]  

(7)

Analytical results for system (2)-(3)-(4) can be found in [2]. In the following section, we will look for non-null stationary solutions for system (2) coupled with boundary and transmission conditions (3) and (4).

### 3. STATIONARY SOLUTIONS

According to equation (2), a stationary solution satisfies the following equations on the arc \( a_i \in \mathcal{A} \) :

\[
\begin{cases}
  v_i x = 0, \\
  \lambda_i^2 u_i x = \alpha_i u_i - v_i, \quad \text{that is to say } v_i = \text{constant}, \\
  u_i = C_i \exp(\alpha_i x / \lambda_i^2) + v_i / \alpha_i.
\end{cases}
\]  

(8)

For simplicity, we explain the computation on a twelve-nodes and twelve-arcs network, represented on the left at Figure 2. The following computation of stationary solutions can be extended in a simple way to more general networks.
Considering the \( v \)-part of system (8) and using boundary condition (3) and condition (6) at each node, we obtain that \( v_i = 0 \), \( 5 \leq i \leq 12 \) and \( v_1 = v_2 = -v_3 = v \), with \( v \) unknown.

Plugging now the expression for \( u \) given by system (8) in the 3 transmission conditions (4) at a node \(^1\), we find that the 13 unknowns left (namely \( v \) and \( \xi_{i,j} \), \( 1 \leq i \leq 12 \)) satisfy a linear system of \( 3 \times 4 = 12 \) equations, whose matrix has a non-null kernel. Unless some particular choices of coefficients \( \xi_{i,j} \), this kernel is of dimension one and, using the mass conservation (7), we find therefore the uniqueness of the non-stationary solution for system (2) with a given initial mass.

4. NUMERICAL SCHEME AND BOUNDARY CONDITIONS

Let us now consider a discretization of system (2) on a network. We denote by \( k \) the time step and by \( t_n = nk \) the discrete times; we also denote by \( h_i \) the space step for discretizing arc \( a_i \in A \) and we consider the discretization points \( x^i_j = jh_i, 0 \leq j \leq M_i + 1 \). The notation \( u_i^{n,j} \) stands for the discretization of function \( u \) on the arc \( a_i \) at time \( t_n \) and at point \( x^i_j \).

We now follow what has been done in [4] for a single interval. Hence, in order to have a valid approximation of the asymptotic behavior of the solutions of system (2), we introduce a finite-differences “asymptotic high-order” scheme which is of higher order near the stationary solutions. However, in the case of a bounded interval, the boundary conditions have to be adapted with respect to the usual ones and we choose them in order to enforce the mass conservation. The same sketch is followed here.

Consequently, on the arc \( a_i \in A \), the numerical scheme, which is asymptotically second order, reads as:

\[

t_i^{n+1,j} - t_i^{n,j} + k \left( \frac{\lambda_i}{h_i} (u_i^{n,j+1} - 2u_i^{n,j} + u_i^{n,j-1}) + \left( \frac{1}{2\lambda_i} - \frac{1}{h_i} \right) (v_i^{n,j+1} - v_i^{n,j-1}) - \frac{\alpha_i}{2\lambda_i} (u_i^{n,j+1} - u_i^{n,j-1}) \right).
\]

\[
v_i^{n+1,j} = v_i^{n,j} + k \left( -\frac{\lambda_i^2}{h_i^2} (u_i^{n,j+1} - u_i^{n,j-1}) + \frac{\lambda_i}{h_i} (v_i^{n,j+1} - 2v_i^{n,j} + v_i^{n,j-1}) - \frac{1}{2} (v_i^{n,j+1} + 2v_i^{n,j} + v_i^{n,j-1}) \right)
\]

\[
+ \frac{\alpha_i}{2} (u_i^{n,j+1} + 2u_i^{n,j} + u_i^{n,j-1}).
\]

To implement this scheme we need four boundary conditions on each arc, namely two boundary conditions by node. The two first boundary conditions are simply given by condition (3) and transmission condition (4). We find the two lacking conditions thanks to the mass conservation. Therefore, we define the total mass as

\[ S_{tot} = \sum_{a_i \in A} S_i^n, \]

\[ S_i^n = h_i \left( u_i^{n,0} + \frac{M_i}{2} u_i^{n,1} + \frac{u_i^{n,M_i+1}}{2} \right) \]

and we compute the boundary conditions in order to cancel \( S_{tot}^{n+1} - S_{tot}^n \).

For an external node \( n_k \in N \) of the arc \( a_i \in A \), we obtain therefore if \( i \in \partial_k \), \( v_i^{n+1,0} = 0 \) and

\[
u_i^{n+1,0} = \left( 1 - \frac{\lambda_i}{h_i} \right) u_i^{n,0} + \frac{\lambda_i}{h_i} u_i^{n,1} + k \left( \frac{1}{2\lambda_i} - \frac{1}{h_i} \right) v_i^{n,1} - \frac{\alpha_i k}{2\lambda_i} (u_i^{n,1} - v_i^{n,0}).
\]

For an internal node \( n_k \in N \) of the arc \( a_i \in A \), we obtain therefore if \( i \in \partial_k \)

\[
u_i^{n+1,0} = \sum_{j \in \text{matheal}_{I_k}} \xi_{i,j} u_i^{n,M_i+1} + \sum_{j \in \text{matheal}_{O_k}} \xi_{i,j} u_i^{n,0},
\]

\[
u_i^{n+1,0} = h_i \left( h_i + \sum_j h_j \xi_{i,j} \right)^{-1} \times \left( u_i^{n,0} \left( 1 - \frac{\lambda_i}{h_i} + \frac{k\alpha_i}{h_i} \right) + u_i^{n,1} \left( 1 - \frac{k}{k\alpha_i} \right) + k u_i^{n,1} \left( 1 - \frac{\alpha_i}{2\lambda_i} \right) \right),
\]

\[
+ k u_i^{n,1} \left( \frac{\lambda_i}{h_i} - \frac{1}{2} \right) \left( 1 - \frac{\alpha_i}{2\lambda_i} \right).
\]

\(^1\) the 4 transmission conditions at a node given by equations (4) count genuinely for 3 equations thanks to relation (5).
5. NUMERICAL RESULTS

In this section, some numerical examples showing the asymptotic behavior of the solutions are reported. Let us consider the network in Fig. 2 composed by twelve nodes and twelve arcs. In such a case, in the internal loop of the network, \( v \) is not null at the equilibrium, as showed in Fig. 1. The asymptotic densities for the internal arcs are depicted in the figure on the right of Fig. 1. A tridimensional representation of the densities on the network at equilibrium is reported in Fig. 2. We notice that the asymptotic solution coincides perfectly with the stationary solution computed in Section 3.

![FIGURE 1](image1.png)

**FIGURE 1.** Stationary solutions for the network composed by 12 nodes and 12 arcs: densities of the 4 arcs in the internal loop at time \( T = 60, \lambda_1 = \lambda = 1, \alpha_i = \alpha = 0.5 \), initial mass \( \mu_0 = 615 \) equally distributed, with \( h_i = h = 0.0004, k = 0.0002 \). On the right the asymptotic flux \( v_1 = v_4 = -v_2 = -v_3 = v = 1.72 \) of the 4 arcs in the internal loop.

![FIGURE 2](image2.png)

**FIGURE 2.** For the same data as in Fig. 1: a 3D visualization of the densities on the whole network.

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