Bio-electric current density imaging via an iterative algorithm with joint sparsity constraints

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Abstract

Neuronal current imaging aims at analyzing the functionality of the human brain through the localization of those regions where the neural current flows. The reconstruction of an electric current distribution from its magnetic field measured by sophisticated superconducting devices in a noninvasive way, gives rise to a highly ill-posed and ill-conditioned inverse problem.

Assuming that each component of the current density vector possesses the same sparse representation with respect to a preassigned multiscale basis, allows us to apply new regularization techniques to the magnetic inverse problem. In particular, we use a joint sparsity constraint as a regularization term and we propose an efficient iterative thresholding algorithm to reconstruct the current distribution. Some bidimensional experiments are presented in order to show the algorithm properties.

Keywords: Magnetoencephalography, Inverse problem, Sparsity constraint, Iterative thresholding, Multiscale basis.
1. Introduction

In neuroscience studies the brain activity can be localized by detecting the regions where neural currents flow. A bioelectric current distribution produces a characteristic magnetic field outside the head which provides information on the processes occurring within the brain. In particular, magnetoencephalographic (MEG) neuroimaging aims at detecting the brain active regions through the measurements by SQUID magnetometers of the tiny magnetic field generated by neuronal currents, \([1, 5, 11]\).

Since the magnetic data do not give an immediate image of the current distribution, we have to solve the associated inverse problem by means of regularization techniques. The Tikhonov regularization with quadratic constraint, gives good results when the quantities under observation are equally distributed in time or space \([6]\). However, the current distribution we want to reconstruct can be often represented as a sum of weighted basic currents with only few significant terms \textit{sparse representation}, hence the current distribution is spatially inhomogeneous. To promote sparsity in the reconstruction of scalar quantities, a regularization technique based on \textit{non quadratic constraints} was introduced in \([4]\) and the solution of the inverse problem was approximated by a convergent thresholded Landweber algorithm, see the papers \([7, 8, 9, 10]\). Numerical tests show that joint sparsity outperforms Tikhonov regularization and soft thresholding, especially when the data are affected by high noise.

The paper is organized as follows. In Section 2 the inverse MEG problem is briefly recalled. Then, in Section 3, we introduce the joint sparsity constraint as a regularization term. A convergent iterative thresholding algorithm which solves the MEG inverse problem is proposed in Section 4. Finally, in Section 5 some numerical tests are presented.

2. The Forward and Inverse Magnetic Problems

Here we refer to a modelization of the head as a set of different regions, say \(V_k\), \(k = 0, \ldots, K\), with constant conductivity \(\sigma_k\), such that \(V_k \subset V_{k+1}, k = 0, \ldots, K-1\). The regions \(V_k, k = 0, \ldots, K\), represent different anatomical parts, i.e. the brain, the cerebrospinal fluid, the meninges, the skull, etc. The neuronal currents \(\vec{J}\) are confined to the brain, represented by the innermost region \(V_0\). In order to obtain a model easily tractable, but still sufficiently realistic, we assume that the volumes \(V_k\) are spheres of increasing radius and centered at the origin.

The external magnetic field \(\vec{B}\) is linked to the current \(\vec{J}\) by the Biot-Savart law \([11]\):

\[
\vec{B}(\vec{r}) = \vec{B}_\infty(\vec{r}) - \frac{\mu_0}{4\pi} \sum_{k=0}^{K} (\sigma_{k+1} - \sigma_k) \int_{\partial V_k} \frac{\Phi(\vec{r}') \vec{e}_k(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d(\partial V_k(\vec{r}')), \tag{1}
\]
where \( \vec{r} \) is a point in the space external to the head and \( \vec{r}' \) is a point in the region \( V_0 \), while \( \vec{B}_\infty \) is the magnetic field in an infinite homogeneous medium with magnetic permeability \( \mu_0 \), i.e.

\[
\vec{B}_\infty(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V_0} \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, d\vec{r}',
\]

(2)

\( \Phi \) is the electric potential on the surfaces \( \partial V_k \) and \( \vec{e}_k \) is the unit normal w.r.t. \( \partial V_k \), the surface between \( V_k \) and \( V_{k+1} \). Note that \( \sigma_{k+1} \) is set equal to 0.

Let \( \vec{q}_l, l = 1, \ldots, N \), be the site coordinates of the magnetometers, which are located on a spherical surface \( \Omega \) centered at the origin. Usually, just the normal component w.r.t. \( \Omega \) is measured, i.e.

\[
B_r(\vec{q}_l) := \vec{B}(\vec{q}_l) \cdot \vec{e}_r(\vec{q}_l), \quad l = 1, \ldots, N,
\]

where \( \vec{e}_r(\vec{q}_l) \) is the unit normal w.r.t. \( \Omega \) in \( \vec{q}_l \).

Here, we assume that the regions \( V_k \) are concentric spheres of increasing radius and centered at the origin. Then, recalling that for any three vectors in \( \mathbb{R}^3 \) it holds \( v \times w \cdot z = -z \times w \cdot v \), one has:

\[
B(\vec{J}, \vec{q}_l) := B_r(\vec{q}_l) = \frac{\mu_0}{4\pi} \int_{V_0} \frac{\vec{e}_r(\vec{q}_l) \times (\vec{r}' - \vec{q}_l)}{|\vec{q}_l - \vec{r}'|^3} \cdot \vec{J}(\vec{r}') \, d\vec{r}'.
\]

(3)

We remark that, due to the spherical symmetry, the magnetic field generated by the electric potential do not contribute to \( B_r \). Unfortunately, there exist silent currents, so that non unique solutions can be expected. Moreover, the magnetic data can be affected by high noise. A regularization mechanism is required both to identify uniquely the solution by taking advantage of possible prior knowledge (in this case the spatial joint-sparsity of the currents) and to remove the noise.

3. The Magnetic Inverse Problem with Sparsity Constraints

We are interested in applications where the current density is spatially inhomogeneous so that it can be represented as a sum of weighted basic currents with only few significant terms. This means that we can assume \( \vec{J} = (J_1, J_2, J_3) \in L_2(V_0; \mathbb{R}^3) \) sparsely represented by a suitable dictionary \( \mathcal{D} := (\psi_\lambda)_{\lambda \in \Lambda} \), i.e.

\[
J_\ell \approx \sum_{\lambda \in \Lambda_S} j_\lambda^\ell \psi_\lambda, \quad j_\lambda^\ell = \langle J_\ell, \psi_\lambda \rangle, \quad \ell = 1, 2, 3,
\]

(4)

where \( \Lambda_S \subset \Lambda \), the small set of significant coefficients, is the same for all the components. As a dictionary we can choose a stable multiscale basis, for instance a wavelet basis [2] or frame.

Thus, the magnetic inverse problem with the sparsity constraint \( \Psi_\mathcal{D} \) consists in minimizing the functional \( J_\Psi(\vec{J}, v) = \Delta(\vec{J}) + \Psi_\mathcal{D}(\vec{J}, v) \), with respect to both \( \vec{J} = (j_\lambda^\ell)_{\lambda \in \Lambda, \ell = 1, 2, 3} \) and an auxiliary weight \( v \), restricted to \( v_\lambda \geq 0 \).
We use as sparsity constraint the joint sparsity measure introduced in [8], i.e.
\[
\Psi_D(\vec{j}, v) = \sum_{\lambda \in \Lambda} v_\lambda \|\vec{j}_\lambda\|_p + \sum_{\lambda \in \Lambda} \omega_\lambda \|\vec{j}_\lambda\|_2^2 + \sum_{\lambda \in \Lambda} \theta_\lambda (\rho_\lambda - v_\lambda)^2,
\]
where \( p \geq 1 \) and \((\theta_\lambda)_{\lambda \in \Lambda}, (\rho_\lambda)_{\lambda \in \Lambda}, (\omega_\lambda)_{\lambda \in \Lambda}\) are positive parameter sequences. Here, \( \| \cdot \|_p \) denotes the usual \( p \)-norm for vectors in \( \mathbb{R}^3 \). In this way, the minimization of \( J_\Psi \) promotes that all the entries of the vector \( \vec{j}_\lambda = (j_{\ell,\lambda})_{\ell=1,2,3} \) have the same sparsity pattern.

By using the decomposition (4), the minimum problem reduces to minimization of the functional
\[
J_\Psi(\vec{j}, v) = \left\| T \vec{j} - M \right\|_R^2 + \Psi_D(\vec{j}, v),
\]
where the entries of \( T \vec{j} \) are given by
\[
(T \vec{j})_l = \sum_{\ell=1}^3 \sum_{\lambda \in \Lambda} j_{\ell,\lambda} \frac{\mu_0}{4\pi} \int_{V_0} \frac{\vec{e}(\vec{q}) \times (\vec{r}' - \vec{q})}{|\vec{r}' - \vec{q}|^3} \psi_\lambda(\vec{r}') d\vec{r}'.
\]

An efficient iterative algorithm to numerically solve this minimum problem will be illustrated in the following Section.

4. An accelerated iterative thresholding algorithm

The minimizer \((\vec{j}^*, v^*)\) of the functional \( J_{\theta,\rho,\omega}^{(p)} \) can be approximated by the following iterative thresholding algorithm deduced from [9].
Algorithm JS

Let $\gamma$ be a suitable relaxation parameter.

Choose the positive sequences $(\theta_\lambda)_{\lambda \in \Lambda}$, $(\rho_\lambda)_{\lambda \in \Lambda}$, $(\omega_\lambda)_{\lambda \in \Lambda}$ and $\tilde{y}^{(0)} = 0$

For $0 \leq k \leq K$ do

Let $\nu^{(0)}_\lambda = \rho_\lambda$ and $\tilde{z}^{(0)}_\lambda = \tilde{y}^{(k)}_\lambda + \gamma \left(T^* (M - T \tilde{y}^{(k)})\right)_\lambda$ $\lambda \in \Lambda$

For $0 \leq r \leq R$ do

$\tilde{z}^{(r+1)}_\lambda = S^{(p)}_{\nu^{(r)}_\lambda} (\tilde{z}^{(r)}_\lambda)$ $\lambda \in \Lambda$

$\nu^{(r+1)}_\lambda = \begin{cases} \rho_\lambda - \frac{1}{4 \theta_\lambda (1 + \omega_\lambda)} \| \tilde{z}^{(r+1)}_\lambda \|_p & \text{if } \| \tilde{z}^{(r+1)}_\lambda \|_p < 2 \theta_\lambda (1 + \omega_\lambda) \rho_\lambda \\ 0 & \text{otherwise} \end{cases}$ $\lambda \in \Lambda$

Approximate $\tilde{y}^{(K+1)}_\lambda \approx \frac{\tilde{z}^{(R+1)}_\lambda}{1 + \omega_\lambda}$ $\lambda \in \Lambda$

Compute $v^{(K+1)}_\lambda = \begin{cases} \rho_\lambda - \frac{1}{4 \theta_\lambda} \| \tilde{y}^{(K+1)}_\lambda \|_p & \text{if } \| \tilde{y}^{(K+1)}_\lambda \|_p < 2 \theta_\lambda \rho_\lambda \\ 0 & \text{otherwise} \end{cases}$ $\lambda \in \Lambda$

We remark that, usually, the values $(v^{(K+1)}_\lambda)_{\lambda \in \Lambda}$ are not needed.

The operator $S^{(p)}_{\nu} (\tilde{z}) = (S^{(p)}_{\nu} (\tilde{z}))_{\ell = 1, 2, 3}$ is a thresholding operator whose explicit expression is reported in [8].

**Theorem 1** Let $p = 1, 2, \infty$, and assume $\inf_{\lambda \in \Lambda} \theta_\lambda \omega_\lambda \geq \frac{4 \kappa_p}{\pi}$, where $\kappa_p$ is a constant whose value is 3 for $p = 1$ and 1 for $p = 2, \infty$. Then, the Algorithm JS converges strongly to the unique pair $(\tilde{y}^*, v^*)$ minimizing the functional $J^{(p)}_{\theta, \rho, \omega}$.

Moreover, we have the error estimate

$$\| \tilde{y}^{(k)} - \tilde{y}^* \|_2 \leq \beta^k \| \tilde{y}^{(0)} - \tilde{y}^* \|_2,$$

where $\beta := \sup_{\lambda \in \Lambda} \frac{4 \theta_\lambda}{4 \theta_\lambda (1 + \omega_\lambda) - \kappa_p} < 1$.

Proof. From [9, Lemma 2.1] it follows that $J^{(p)}_{\theta, \rho, \omega}$ is strictly convex and has a unique minimizer. Then, the claim follows from Prop. 3.4 and Th. 4.3 in [9].

In order to implement the algorithm we need to evaluate $T^* T \tilde{y}$, whose explicit
expression is given in [7]. Let $\mathcal{M}$ be the matrix whose entries are the coordinates of $T^*T$ in the multiscale basis $(\psi_\lambda)_{\lambda \in \Lambda}$, i.e.

$$\mathcal{M}(\lambda, \ell, (\mu, m)) := \sum_{l=1}^{N} (A_{\ell,l} \psi_\lambda)(A_{m,l} \psi_\mu), \quad \lambda, \mu \in \Lambda, \quad \ell, m = 1, 2, 3,$$

where the operator $A_{\ell,l} : L^2(V_0; \mathbb{R}) \to \mathbb{R}$ is defined as

$$A_{\ell,l} f := \frac{\mu_0}{4\pi} \int_{V_0} \frac{\bar{e}_z(\bar{q}_l) \times (\bar{q}_l - \bar{r}')}{|\bar{q}_l - \bar{r}'|^3} f(\bar{r}') d\bar{r}', \quad l = 1, \ldots, N.$$ 

Then, we obtain the following expression:

$$\left(T^*T \hat{j}\right)_\lambda, \ell = \sum_\mu \sum_{m=1}^{3} j^m_\mu \mathcal{M}(\lambda, \ell, (\mu, m)), \quad \lambda \in \Lambda, \quad \ell = 1, 2, 3. \quad (10)$$

Since $\mathcal{M}$ is a bi-infinite matrix, in order to implement an efficient procedure to compute $T^*T \hat{j}$ we need the amplitude of the entries of $\mathcal{M}$ to decay fast when $\lambda$ and $\mu$ increase. As a multiscale dictionary we choose Daubechies wavelet basis with $d = 4$ vanishing moments and $\Omega_\lambda := \text{supp} (\psi_\lambda) \sim 2^{-|\lambda|}$, with $|\lambda|$ the spatial resolution scale of $\psi_\lambda$. Moreover, the basis functions have a prescribed smoothness, and fast decay, i.e. $|\psi_\lambda| \leq C 2^{-3/2|\lambda|}$. It can be shown that $\mathcal{M}$ has compressibility properties w.r.t. such a basis (cf. [7]), so that $T^*T \hat{j}$ can be evaluated efficiently.

5. A bidimensional test

In the numerical tests the magnetic data are generated by three horizontal bidimensional current dipoles located in the plane $\Pi_0 = \{x, y \in \mathbb{R}, z = 0\}$. The magnetic field is sampled by 400 magnetometers located on a regular horizontal grid at height $\delta = 1$. Note that we use adimensional measure units in the tests.

This setting can be used to model both the problem of localizing shallow neural sources and non destructive testing of thin structures. In fact, the current dipoles can be viewed both as sources of brain activity and as discontinuities in a given current distribution. We remark that in the bidimensional case the inverse problem has a unique solution, nevertheless, it can be ill-conditioned for the presence of noise. We discretize the plane $\Pi_0$ with 32 pixels for each dimension, then 2 multiscale levels are used for the current decomposition. We performed some preliminary tests on Algorithm JS.

Figures 2 and 3 are the current images reconstructed from the magnetic field, depicted, respectively, on the left and on the right of Fig. 1. In both cases a comparison is established between uncoupled soft-thresholding, i.e. Algorithm JS with $p = 1$ and $R = 0$ (left) and joint-thresholding, i.e. Algorithm JS with $p = 2$ and $R > 0$, (right). Here we have set $\theta_\lambda = 10^{-4}$ and $\omega_\lambda = 10^{-3}$ for each $\lambda \in \Lambda$ and $\rho_\lambda = 10^{-3}$. In Figures 5 and 6 are represented the current
Figure 1: The magnetic field produced by three current dipoles located in $(-0.5, -0.4, 0), (-0.1, 0.1, 0), (0.3, 0.475, 0)$ (left) and its perturbation by Gaussian noise (right). The black points represent the magnetometer sites.

Figure 2: The current intensity reconstructed starting from the magnetic field displayed in Fig. 1 (left) by using 100 iterations of, respectively, soft-thresholding (left) and joint-thresholding (right) with $\rho_\lambda = 10^{-3}$.

images reconstructed from the magnetic field, depicted, respectively, on the left and on the right of Fig. 4. In both cases a comparison is proposed between uncoupled soft-thresholding (left) and joint-thresholding (right) and we have set $\theta_\lambda = \omega_\lambda = 10^{-4}$ for each $\lambda \in \Lambda$ and $\rho_\lambda = 10^{-2}$.

As a comparison, we now show localization results obtained by means of quadratic Tikhonov regularization in the case when the data are distorted by high noise. In real fact, quadratic Tikhonov regularization is not able to give a satisfactory localization of the current sources.

In particular, in Figures 7 and 8, the current distribution reconstructed starting from the magnetic fields displayed, respectively, on the right of Fig. 1 and Fig. 4, is depicted. In particular, by using the quadratic Tikhonov regularization for two different values of the regularization parameter, namely $w = 10^{-3}$ and $w = 552$ (from the discrepancy principle) is displayed.

Notice that in both cases, when the regularization parameter is low (equal to $10^{-3}$ in the example) the current is not well reconstructed, while a greater
Figure 3: The current intensity reconstructed starting from the noisy magnetic field displayed in Fig. 1 (right) by using 100 iterations of, respectively, soft-thresholding (left) and joint-thresholding (right) with $\rho_\lambda = 10^{-3}$.

Figure 4: The magnetic field produced by three current dipoles located in $(-0.2, 0.1, 0), (0.0, 0.5, 0), (0.2, 0.2, 0)$ (left) and its perturbation by Gaussian noise (right). The black points represent the magnetometer sites.

Figure 5: The current intensity reconstructed starting from the magnetic field displayed in Fig. 4 (left) by using 100 iterations of, respectively, soft-thresholding (left) and joint-thresholding (right) with $\rho_\lambda = 10^{-2}$. 
Figure 6: The current intensity reconstructed starting from the noisy magnetic field displayed in Fig. 4 (right) by using 100 iterations of, respectively, soft-thresholding (left) and joint-thresholding (right) with $\rho_\lambda = 10^{-2}$.

Figure 7: The current distribution reconstructed by using the quadratic Tikhonov regularization starting from the noisy magnetic field displayed in Fig. 1. Two different values of the regularization parameter have been used: $w = 10^{-3}$ (left) and $w = 552$ (right).

Figure 8: The current distribution reconstructed by using the quadratic Tikhonov regularization starting from the noisy magnetic field displayed in Fig. 4. Two different values of the regularization parameter have been used: $w = 10^{-3}$ (left) and $w = 429$ (right).
regularization parameter blurs the image (Fig. 7 and Fig. 8 on the right, where the regularization parameter is chosen by means of the discrepancy principle).

References


