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Numerical Schemes for Conservation Laws: a Guide for a Quick Implementation. Part I

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These notes list some numerical methods for the advection equation with constant coefficients in one dimension.

Contents

1 Basic notations

Let us consider a structured (rectangular) grid $\{x_i, t^n\}$, $i = 1, \dots, N$, $n = 1, \dots, M$, with time step Δt and space step Δx. Denote by U_i^n the approximate value of a function u at the grid node (x_i, t^n).
Let us define the constant

$$\lambda := \frac{\Delta t}{\Delta x}.$$

Let us consider the initial value problem

$$\begin{cases} \partial_t u(x,t) + \partial_x[f(u(x,t))] = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x). \end{cases}$$

Hereafter we will consider numerical schemes written in the form

$$\partial_t u(x_i, t^n) = \mathcal{L}_i(u(\cdot, t^n)),$$

so that one can easily decouple the space discretization and the time discretization.

2 Time discretization

Given $\mathcal{L}_i$, one can write a high-order-in-time approximation scheme by using the following approximations [7, Sect. 4.2.1].

2.1 Forward Euler I order

$$U^{n+1} = U_i^n + \Delta t \mathcal{L}(U^n)$$

2.2 Runge-Kutta TVD II order

$$U^{(1)} = U^n + \Delta t \mathcal{L}(U^n)$$
$$U^{n+1} = \frac{1}{2}U^n + \frac{1}{2}U^{(1)} + \frac{1}{2}\Delta t \mathcal{L}(U^{(1)})$$

2.3 Runge-Kutta TVD III order

$$U^{(1)} = U^n + \Delta t \mathcal{L}(U^n)$$
$$U^{(2)} = \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}\Delta t \mathcal{L}(U^{(1)})$$
$$U^{n+1} = \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}\Delta t \mathcal{L}(U^{(2)})$$

3 Space discretization for the advection equation with constant coefficient

Let us consider the initial value problem

$$\begin{cases} \partial_t u(x,t) + a\partial_x u(x,t) = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), \end{cases}$$

with $a > 0$. Let us define the CFL parameter

$$\zeta := a\lambda$$

and assume that $\zeta \leq 1$ (included semi-Lagrangian schemes).

3.1 Upwind method (`UPW`) [4]

$$\Delta t \mathcal{L}_i(U^n) = -\zeta(U_i^n - U_{i-1}^n), \qquad i = 2, \dots, N$$

3.2 Lax-Wendroff (`LW`) [4]

$$\Delta t \mathcal{L}_i(U^n) = -\frac{1}{2}\zeta(U_{i+1}^n - U_{i-1}^n) + \frac{1}{2}\zeta^2(U_{i+1}^n - 2U_i^n + U_{i-1}^n), \qquad i = 2, \dots, N-1$$

3.3 Beam Warming (`BW`) [4]

$$\Delta t \mathcal{L}_i(U^n) = -\frac{1}{2}\zeta(3U_i^n - 4U_{i-1}^n + U_{i-2}^n) + \frac{1}{2}\zeta^2(U_i^n - 2U_{i-1}^n + U_{i-2}^n), \qquad i = 3, \dots, N$$

3.4 Semi-Lagrangian I order (`SL1`) [3]

$$\Delta t \mathcal{L}_i(U^n) = -U_i^n + I[U^n](x^*) \qquad i = 2, \dots, N$$
$$x^* = x_i - a\Delta t, \qquad (x^* \in [x_{i-1}, x_i))$$
$$I[U^n](x) = \gamma_1 x + \gamma_2, \quad x \in \mathbb{R}$$

$\gamma \in \mathbb{R}^2$ solution to

$$\begin{pmatrix} x_{i-1} & 1 \\ x_i & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} U_{i-1}^n \\ U_i^n \end{pmatrix}$$

3.5 Semi-Lagrangian II order (SL2) [3]

$$\begin{aligned}
&\Delta t \mathcal{L}_i(U^n) = -U_i^n + I[U^n](x^*) \qquad i = 3, \ldots, N \\
&x^* = x_i - a\Delta t, \qquad (x^* \in [x_{i-1}, x_i)) \\
&I[U^n](x) = \gamma_1 x^2 + \gamma_2 x + \gamma_3, \quad x \in \mathbb{R} \\
&\gamma \in \mathbb{R}^3 \text{ solution to} \\
&\begin{pmatrix} x_{i-2}^2 & x_{i-2} & 1 \\ x_{i-1}^2 & x_{i-1} & 1 \\ x_i^2 & x_i & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} U_{i-2}^n \\ U_{i-1}^n \\ U_i^n \end{pmatrix}
\end{aligned}$$

3.6 Semi-Lagrangian III order (SL3) [3]

$$\begin{aligned}
&\Delta t \mathcal{L}_i(U^n) = -U_i^n + I[U^n](x^*) \qquad i = 3, \ldots, N-1 \\
&x^* = x_i - a\Delta t, \qquad (x^* \in [x_{i-1}, x_i)) \\
&I[U^n](x) = \gamma_1 x^3 + \gamma_2 x^2 + \gamma_3 x + \gamma_4, \quad x \in \mathbb{R} \\
&\gamma \in \mathbb{R}^4 \text{ solution to} \\
&\begin{pmatrix} x_{i-2}^3 & x_{i-2}^2 & x_{i-2} & 1 \\ x_{i-1}^3 & x_{i-1}^2 & x_{i-1} & 1 \\ x_i^3 & x_i^2 & x_i & 1 \\ x_{i+1}^3 & x_{i+1}^2 & x_{i+1} & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} = \begin{pmatrix} U_{i-2}^n \\ U_{i-1}^n \\ U_i^n \\ U_{i+1}^n \end{pmatrix}
\end{aligned}$$

3.7 Flux Limiter (FLUXLIM) [4]

$$\begin{aligned}
&\Delta t \mathcal{L}_i(U^n) = \lambda(H^- - H^+), \qquad i = 3, \ldots, N-1 \\
&H^+ = aU_i^n + \frac{1}{2}a(1-\zeta)\phi\left(\frac{U_i^n - U_{i-1}^n}{U_{i+1}^n - U_i^n + \varepsilon}\right)(U_{i+1}^n - U_i^n) \\
&H^- = aU_{i-1}^n + \frac{1}{2}a(1-\zeta)\phi\left(\frac{U_{i-1}^n - U_{i-2}^n}{U_i^n - U_{i-1}^n + \varepsilon}\right)(U_i^n - U_{i-1}^n) \\
&\varepsilon = \text{small constant } (10^{-6}) \\
&\phi(r) = \begin{cases} \text{minmod}(1, r) & \text{[minmod limiter]} \\ \max(0, \min(1, 2r), \min(2, r)) & \text{[superbee limiter]} \end{cases} \\
&\text{minmod}(a, b) = \begin{cases} a & \text{if } |a| \le |b| \text{ and } ab > 0 \\ b & \text{if } |a| > |b| \text{ and } ab > 0 \\ 0 & \text{if } ab \le 0 \end{cases}
\end{aligned}$$

3.8 Semi-Lagrangian II order WENO (SL2WENO) [1, 3]

$$
\begin{aligned}
&\Delta t \mathcal{L}_i(U^n) = -U_i^n + I[U^n](x^*) \qquad i = 3, \dots, N-2 \\
&x^* = x_i - a\Delta t, \qquad (x^* \in [x_{i-1}, x_i)) \\
&I[U^n](x) = w_L P_L(x) + w_R P_R(x), \quad x \in \mathbb{R} \\
&P_L(x) = \gamma_1 x^2 + \gamma_2 x + \gamma_3, \quad x \in \mathbb{R} \\
&P_R(x) = \delta_1 x^2 + \delta_2 x + \delta_3, \quad x \in \mathbb{R} \\
&\begin{pmatrix} x_{i-2}^2 & x_{i-2} & 1 \\ x_{i-1}^2 & x_{i-1} & 1 \\ x_i^2 & x_i & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} U_{i-2}^n \\ U_{i-1}^n \\ U_i^n \end{pmatrix} \\
&\begin{pmatrix} x_{i-1}^2 & x_{i-1} & 1 \\ x_i^2 & x_i & 1 \\ x_{i+1}^2 & x_{i+1} & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} U_{i-1}^n \\ U_i^n \\ U_{i+1}^n \end{pmatrix} \\
&w_L(x) = \frac{\alpha_L(x)}{\alpha_L(x) + \alpha_R(x)}, \quad x \in \mathbb{R}, \qquad\qquad w_R(x) = \frac{\alpha_R(x)}{\alpha_L(x) + \alpha_R(x)}, \quad x \in \mathbb{R} \\
&\alpha_L(x) = \frac{C_L(x)}{(\beta_L + \varepsilon)^2}, \quad x \in \mathbb{R} \qquad\qquad \alpha_R(x) = \frac{C_R(x)}{(\beta_R + \varepsilon)^2}, \quad x \in \mathbb{R} \\
&\varepsilon = \text{small constant } (10^{-6}) \\
&\beta_L = \frac{13}{12}(U_{i-2}^n)^2 + \frac{16}{3}(U_{i-1}^n)^2 + \frac{25}{12}(U_i^n)^2 - \frac{13}{3}U_{i-2}^n U_{i-1}^n + \frac{13}{6}U_{i-2}^n U_i^n - \frac{19}{3}U_i^n U_{i-1}^n \\
&\beta_R = \frac{13}{12}(U_{i+1}^n)^2 + \frac{16}{3}(U_i^n)^2 + \frac{25}{12}(U_{i-1}^n)^2 - \frac{13}{3}U_{i+1}^n U_i^n + \frac{13}{6}U_{i+1}^n U_{i-1}^n - \frac{19}{3}U_i^n U_{i-1}^n \\
&C_L(x) = \frac{x_{i+1} - x}{3\Delta x}, \qquad C_R(x) = \frac{x - x_{i-2}}{3\Delta x}
\end{aligned}
$$

3.9 Conservative Semi-Lagrangian III order (CSL3) [6]

$$
\begin{aligned}
&\Delta t \mathcal{L}_i(U^n) = -\frac{1}{\Delta x}(H^+ - H^-), \qquad i = 3, \dots, N-1 \\
&H^+ = \Delta x(C_1 U_{i-1}^n + C_2 U_i^n + C_3 U_{i+1}^n) \\
&H^- = \Delta x(C_1 U_{i-2}^n + C_2 U_{i-1}^n + C_3 U_i^n) \\
&C_1 = \frac{1}{6}(\zeta^3 - \zeta), \quad C_2 = -\frac{1}{3}\zeta^3 + \frac{1}{2}\zeta^2 + \frac{5}{6}\zeta, \quad C_3 = \frac{1}{6}\zeta^3 - \frac{1}{2}\zeta^2 + \frac{1}{3}\zeta
\end{aligned}
$$

3.10 Conservative Semi-Lagrangian V order (CSL5) [6]

$$
\begin{aligned}
&\Delta t\mathcal{L}_i(U^n) = -\frac{1}{\Delta x}(H^+ - H^-), \qquad i = 4,\dots,N-2\\
&H^+ = \Delta x(C_1U^n_{i-2} + C_2U^n_{i-1} + C_3U^n_i + C_4U^n_{i+1} + C_5U^n_{i+2})\\
&H^- = \Delta x(C_1U^n_{i-3} + C_2U^n_{i-2} + C_3U^n_{i-1} + C_4U^n_i + C_5U^n_{i+1})\\
&C_1 = \frac{1}{30}\zeta - \frac{1}{24}\zeta^3 + \frac{1}{120}\zeta^5, \quad C_2 = -\frac{13}{60}\zeta - \frac{1}{24}\zeta^2 + \frac{1}{4}\zeta^3 + \frac{1}{24}\zeta^4 - \frac{1}{30}\zeta^5\\
&C_3 = \frac{47}{60}\zeta + \frac{5}{8}\zeta^2 - \frac{1}{3}\zeta^3 - \frac{1}{8}\zeta^4 + \frac{1}{20}\zeta^5, \quad C_4 = \frac{9}{20}\zeta - \frac{5}{8}\zeta^2 + \frac{1}{12}\zeta^3 + \frac{1}{8}\zeta^4 - \frac{1}{30}\zeta^5\\
&C_5 = -\frac{1}{20}\zeta + \frac{1}{24}\zeta^2 + \frac{1}{24}\zeta^3 - \frac{1}{24}\zeta^4 + \frac{1}{120}\zeta^5
\end{aligned}
$$

3.11 Conservative Semi-Lagrangian III order WENO (CSL3WENO) [6]

$$
\begin{aligned}
&\Delta t\mathcal{L}_i(U^n) = -\frac{1}{\Delta x}(H^+ - H^-), \qquad i = 3,\dots,N-1\\
&H^+ = w_1^+\mathcal{H}_1^+ + w_2^+\mathcal{H}_2^+, \qquad H^- = w_1^-\mathcal{H}_1^- + w_2^-\mathcal{H}_2^-\\
&w_i^\pm = \frac{\widetilde{w}_i^\pm}{\widetilde{w}_1^\pm + \widetilde{w}_2^\pm}, \quad i = 1,2\\
&\widetilde{w}_i^\pm = \frac{\gamma_i}{(\varepsilon + \beta_i^\pm)^2}, \quad i = 1,2\\
&\varepsilon = \text{small constant } (10^{-6})\\
&\gamma_1 = \frac{1}{3}(1+\zeta), \qquad \gamma_2 = \frac{1}{3}(2-\zeta)\\
&\beta_1^+ = (U^n_{i-1} - U^n_i)^2, \quad \beta_2^+ = (U^n_i - U^n_{i+1})^2, \quad \beta_1^- = (U^n_{i-2} - U^n_{i-1})^2, \quad \beta_2^- = (U^n_{i-1} - U^n_i)^2\\
&\mathcal{H}_1^+ = \Delta x(C_1U^n_{i-1} + C_2U^n_i), \qquad \mathcal{H}_2^+ = \Delta x(C_3U^n_i + C_4U^n_{i+1})\\
&\mathcal{H}_1^- = \Delta x(C_1U^n_{i-2} + C_2U^n_{i-1}), \qquad \mathcal{H}_2^- = \Delta x(C_3U^n_{i-1} + C_4U^n_i)\\
&C_1 = -\frac{1}{2}\zeta + \frac{1}{2}\zeta^2, \quad C_2 = -\frac{1}{2}\zeta^2 + \frac{3}{2}\zeta, \quad C_3 = \frac{1}{2}\zeta + \frac{1}{2}\zeta^2, \quad C_4 = -\frac{1}{2}\zeta^2 + \frac{1}{2}\zeta
\end{aligned}
$$

3.12 WENO III order (WENO3) [5]

$$\mathcal{L}_i(U^n) = -\frac{a}{\Delta x}\Big(R_i(x_{i+1/2}) - R_{i-1}(x_{i-1/2})\Big)$$
$$R_i(x) = w_i P_i^-(x) + (1-w_i)P_i^+(x), \quad x \in \mathbb{R}$$
$$R_{i-1}(x) = w_{i-1} P_{i-1}^-(x) + (1-w_{i-1})P_{i-1}^+(x), \quad x \in \mathbb{R}$$
$$w_i = \frac{\frac{1}{2(\varepsilon+\beta_i)^2}}{\frac{1}{2(\varepsilon+\beta_i)^2} + \frac{1}{(\varepsilon+\beta_{i+1})^2}} = \left(1 + \frac{2(\varepsilon+\beta_i)^2}{(\varepsilon+\beta_{i+1})^2}\right)^{-1}$$
$$w_{i-1} = \left(1 + \frac{2(\varepsilon+\beta_{i-1})^2}{(\varepsilon+\beta_i)^2}\right)^{-1}$$
$$\varepsilon = \text{small constant } (10^{-6})$$
$$\beta_{i-1} = (U_{i-1}^n - U_{i-2}^n)^2, \quad \beta_i = (U_i^n - U_{i-1}^n)^2, \quad \beta_{i+1} = (U_{i+1}^n - U_i^n)^2$$
$$P_i^-(x) = U_i^n + \frac{U_i^n - U_{i-1}^n}{\Delta x}(x - x_i), \quad x \in \mathbb{R}$$
$$P_i^+(x) = U_i^n + \frac{U_{i+1}^n - U_i^n}{\Delta x}(x - x_i), \quad x \in \mathbb{R}$$
$$P_{i-1}^-(x) = U_{i-1}^n + \frac{U_{i-1}^n - U_{i-2}^n}{\Delta x}(x - x_{i-1}), \quad x \in \mathbb{R}$$
$$P_{i-1}^-(x) = U_{i-1}^n + \frac{U_i^n - U_{i-1}^n}{\Delta x}(x - x_{i-1}), \quad x \in \mathbb{R}$$

3.13 Runge Kutta Discontinuous Galerkin III order (RKDG3) [2]

$$U(x,t^{n+1}) = V_{0,i}^{n+1} + \frac{12}{\Delta x}V_{1,i}^{n+1}(x-x_i) + \frac{180}{\Delta x^2}V_{2,i}^{n+1}\left((x-x_i)^2 - \frac{\Delta x^2}{12}\right), \quad x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

or for the mean value

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U(x,t^{n+1})dx = \frac{1}{12}\left(V_{0,i}^{n+1} - 6V_{1,i}^{n+1} + 30V_{2,1}^{n+1}\right) + \frac{5}{12}\left(V_{0,i}^{n+1} - \frac{6}{\sqrt{5}}V_{1,i}^{n+1} - 6V_{2,1}^{n+1}\right)$$
$$+\frac{5}{12}\left(V_{0,i}^{n+1} + \frac{6}{\sqrt{5}}V_{1,i}^{n+1} - 6V_{2,1}^{n+1}\right) + \frac{1}{12}\left(V_{0,i}^{n+1} + 6V_{1,i}^{n+1} + 30V_{2,i}^{n+1}\right)$$

Initialization:

$$x_1 = x_i - \frac{\Delta x}{2}, \quad x_2 = x_i - \frac{\Delta x}{2}\sqrt{\frac{1}{5}}, \quad x_3 = x_i + \frac{\Delta x}{2}\sqrt{\frac{1}{5}}, \quad x_4 = x_i + \frac{\Delta x}{2}$$

$$V_0^0 = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(x)dx = \frac{1}{12}u_0(x_1) + \frac{5}{12}u_0(x_2) + \frac{5}{12}u_0(x_3) + \frac{1}{12}u_0(x_4)$$

$$V_1^0 = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (x - x_i)u_0(x)dx = \frac{1}{12\Delta x}(x_1 - x_i)u_0(x_1) + \frac{5}{12\Delta x}(x_2 - x_i)u_0(x_2)$$

$$+ \frac{5}{12\Delta x}(x_3 - x_i)u_0(x_3) + \frac{1}{12\Delta x}(x_4 - x_i)u_0(x_4)$$

$$V_2^0 = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left((x - x_i)^2 - \frac{\Delta x^2}{12}\right) u_0(x)dx = \frac{1}{12\Delta x^2}\left((x_1 - x_i)^2 - \frac{\Delta x^2}{12}\right) u_0(x_1)$$

$$+ \frac{5}{12\Delta x^2}\left((x_2 - x_i)^2 - \frac{\Delta x^2}{12}\right) u_0(x_2) + \frac{5}{12\Delta x^2}\left((x_3 - x_i)^2 - \frac{\Delta x^2}{12}\right) u_0(x_3)$$

$$+ \frac{1}{12\Delta x^2}\left((x_4 - x_i)^2 - \frac{\Delta x^2}{12}\right) u_0(x_4)$$

Time iteration:

$$V_l^{n+1} = \frac{1}{3}V_l^n + \frac{2}{3}V_l^{(2)} + \frac{2}{3}\Delta t\,\mathcal{L}_l(V_0^{(2)}, V_1^{(2)}, V_2^{(2)}), \qquad l = 0, 1, 2$$

$$V_l^{(2)} = \frac{3}{4}V_l^n + \frac{1}{4}V_l^{(1)} + \frac{1}{4}\Delta t\,\mathcal{L}_l(V_0^{(1)}, V_1^{(1)}, V_2^{(1)}), \qquad l = 0, 1, 2$$

$$V_l^{(1)} = V_l^n + \Delta t\,\mathcal{L}_l(V_0^n, V_1^n, V_2^n), \qquad l = 0, 1, 2$$

Space discretization:

$$\mathcal{L}_{0,i}(U) = -\frac{1}{\Delta x}(F_i^+ - F_i^-), \quad i = 2, \ldots, N-1$$

$$\mathcal{L}_{1,i}(U) = -\frac{1}{2\Delta x}(F_i^+ + F_i^-) + \frac{1}{12\Delta x}(f_0^i + 5f_1^i + 5f_2^i + f_3^i)$$

$$\mathcal{L}_{2,i}(U) = -\frac{1}{6\Delta x}(F_i^+ - F_i^-) + \frac{1}{12\Delta x}(f_3^i - f_0^i - \sqrt{5}f_1^i + \sqrt{5}f_2^i + f_3^i)$$

$$f_0^i = a\left(V_{0,i} - 6V_{1,i} + 30V_{2,i}\right), \qquad f_1^i = a\left(V_{0,i} - \frac{6}{\sqrt{5}}V_{1,i} - 6V_{2,i}\right)$$

$$f_2^i = a\left(V_{0,i} + \frac{6}{\sqrt{5}}V_{1,i} - 6V_{2,i}\right), \qquad f_3^i = a\left(V_{0,i} + 6V_{1,i} + 30V_{2,i}\right)$$

$$F^- = F_{num}(U^-_{i-\frac{1}{2}}, U^+_{i-\frac{1}{2}}) = a\,U^-_{i-\frac{1}{2}}, \qquad F^+ = F_{num}(U^-_{i+\frac{1}{2}}, U^+_{i+\frac{1}{2}}) = a\,U^-_{i+\frac{1}{2}}$$

$$U^-_{i-\frac{1}{2}} = V_{0,i-1} + 6V_{1,i-1} + 30V_{2,i-1}, \qquad U^+_{i-\frac{1}{2}} = V_{0,i} - 6V_{1,i} + 30V_{2,i}$$

$$U^-_{i+1/2} = V_{0,i} + 6V_{1,i} + 30V_{2,i}, \qquad U^+_{i+1/2} = V_{0,i+1} - 6V_{1,i+1} + 30V_{2,i+1}$$

N.B. CFL condition requires $\zeta < \frac{1}{6}$.
N.B. This version does not include slope limiters.

General remarks

In the case of constant advection with CFL condition $\zeta \leq 1$, we have the following equivalences: SL1≡UPW, SL2≡BW, SL3≡CSL3.

References

[1] E. Carlini, R. Ferretti, G. Russo, *A weighted essentially nonoscillatory, large time-step scheme for Hamilton-Jacobi equations*, SIAM J. Sci. Comput., 27 (2005), 1071–1091.

[2] B. Cockburn, C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation law: general framework*, Mathematics of Computation, 52 (1989), 411–435.

[3] M. Falcone, R. Ferretti, *Semi-Lagrangian approximation schemes for linear and Hamilton–Jacobi equations*, SIAM, 2014.

[4] R. J. LeVeque, *Numerical methods for conservation laws*, Birkhäuser, 1992.

[5] X-D. Liu, S. Osher, T. Chan, *Weighted essentially non-oscillatory schemes*, Journal of Computational Physics, 115 (1994), 200–212.

[6] J.-M. Qiu, C.-W. Shu, *Conservative high order semi-Lagrangian finite difference WENO methods for advection in incompressible flow*, Journal of Computational Physics, 230 (2011), 863–889.

[7] C.-W. Shu, *Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws*, NASA/CR-97-206253, ICASE Report No. 97-65, 1997.