Numerical methods for option pricing in jump-diffusion markets

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To the other 4+1 Briani

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Introduction

The aim of this Thesis is to investigate numerical approximations of solutions to integro-differential problems arising from derivatives pricing in a market driven by jump-diffusion processes.

The pricing of financial instruments by numerical methods has become a fundamental component in the set of techniques available to practitioners of modern quantitative finance. Starting from the paper by Black and Scholes [14], under suitable conditions, the price of financial derivatives may be related to the solution of Partial Differential Equations (PDE). Moreover, pricing in sophisticate and more realistic models (such as, pricing in incomplete markets, large investor economy, jump-diffusion models, pricing exotic options), leads to the solution of nonlinear parabolic problems, possibly degenerate. In general, problems of these types cannot be solved in closed form. Therefore, a numerical evaluation of the solutions is required.

In this Thesis, we study the analytical numerical framework for convergence results and we apply high-order finite differences efficient approximations to parabolic differential problems. Special attention is given to parabolic integro-differential equations, which arise when considering jump-diffusion processes.

Financial motivations

The standard approach to the valuation of derivatives is based on a suitable specification of a stochastic process for the underlying asset. Historically, the protagonist role in describing the evolution of the market prices has been played by the continuous diffusion process. Unfortunately, empirical events (as for example the market crash of 1987) showed flaws in this model: the real data seems to be not log-normal distributed. Many empirical studies suggest that the real distributions have a higher peak and two heavier tails than those of the normal distribution. In addition, a Brownian evolution implies continuity of the market evolution, which does not reflect real markets. Analyzing financial data it comes out that the prices process can jump (see for instance [18]).

More general models for the stochastic dynamics have then been proposed to face these problems: stochastic volatility functions [33, 34], jump-diffusion models [50], and general singular Lévy models [19] are the most recent ones. In this Thesis, we
shall focus our attention on the jump-diffusion model, first proposed by Merton [50]:
he proposed to add to the continuous model a jump component, where the jumps are counted by a Poisson process and the jump amplitude is log-normal distributed.

Once an underlying model has been given, more hypothesis on the market have to be introduced to find the fair price for contingent claims.
A need for any trader in the market is to cover his investment against the risk of the market (hedge from the risk), which is given by the random sources of the model.
Black and Scholes [14], described the market by a diffusion model and introduced the key concept of dynamic hedging, whereby a way to hedge from the risk is to invest in the market products in a suitable way.
The two basic assumptions of the Black and Scholes model were the absence of arbitrage and the completeness of the market. In a complete Black and Scholes market, the principle of absence of arbitrage leads to a uniquely defined price, that is given in terms of an expectation value with respect to the unique equivalent martingale (risk neutral) measure. Combined with Ito’s formula and Feynman-Kac theorem (see for instance [39]), such expectation is related to the solution of a parabolic partial differential equation. Moreover, this setting allows to derive an explicit formula for the price of European call and put options.

With the introduction of jumps, the market becomes incomplete (the random sources of the model becomes two, while the product traded is still one) and the derivation of the fair price of derivatives becomes somewhat more involved. Once, this difficulty is overcome, i.e. a definition of the risk has been done and a way to hedge from it has been selected (see for instance [36, 37, 38, 60, 59, 20]), by the generalized Ito’s calculus [39] it is still possible to characterize the fair price of derivatives as the solution to a partial integro-differential problem of the form,

\[
\begin{align*}
\frac{\partial u}{\partial t} - \mathcal{L}_T(x, t, \mathcal{I}, \mathcal{D}, \mathcal{D}^2)u + H(x, t, \mathcal{D}u, \mathcal{I}u) &= 0, \\
u(x, 0) &= u_0(x),
\end{align*}
\] (0.0.1)

where \(u_0\) is a continuous initial data, \(\mathcal{L}_T\) is a linear degenerate elliptic operator and \(H\) is a nonlinear first order operator. Here \(\mathcal{I}u\) is an integral term given by

\[
\mathcal{I}u = \int_{\mathbb{R}^D} M(u(x + z, t), u(x, t)) \mu_{x,t}(dz),
\] (0.0.2)

where \(\mu_{x,t}\) is a positive bounded measure and \(M\) is a function which is non decreasing in the first argument, \(M(u, u) = 0\) and such that

\[
M(u, v) - M(w, z) \leq c((u - w)_+ + |v - z|).
\]

The integral term comes out to take into account the variations of the price as a result of the jumps in the underlying process, over all possible values of the jumps amplitudes.
For the specific model by Merton, an analytical formula for the price of the European call and put options may be given as an infinite sum \([50]\). This is a particular lucky case, and this formula is still not in a proper closed form. Therefore, we emphasize that a numerical evaluation of the solution of problem (0.0.1) is needed.

### Analytical framework and convergence

From an analytical point of view, the problem of existence and uniqueness of solutions of (0.0.1) has been intensively studied in the case of uniformly parabolic problems, using the tools of Green functions and Sobolev spaces \([29]\). Unfortunately, due to the possible degeneracy of the parabolic operator, which is typical of incomplete markets, see for instance \([48, 5, 3]\), and the presence of possible nonlinear terms, as in a large investor economy, see for instance \([5, 3]\), we need to study weaker (not continuously differentiable) solutions. A natural approach is given by viscosity solutions \([23]\), which extend the classical notion of differentiable solutions, without changing the comparisons properties: they are weak enough to ensure existence, still assuring the uniqueness of the solution.

Starting from \([24]\), the numerical approximation of viscosity solutions has been widely studied. For second order problems let us refer to the fundamental paper by Barles and Souganidis \([11]\), who first showed convergence results for a large class of numerical schemes to the solution of fully nonlinear second order elliptic or parabolic PDE.

In the viscosity solutions framework, integro-differential problems have been first studied in \([1]\). More general and recent results have been proven in \([5, 3, 2]\). In this Thesis, we provide the very first analysis of the numerical approximation of viscosity solutions of integro-differential problems. We prove that a numerical scheme for well-posed problems of the form (0.0.1), which satisfies the classical assumptions of monotonicity, stability and consistency, and moreover, it is monotone with respect the integral approximation, yields an approximation which converges uniformly to the unique viscosity solution of the continuous problem.

### High-order approximations

Once a convergence framework has been established, to effectively compute the numerical solutions, we still need to introduce and analyze efficient schemes. In the framework of numerical approximations of linear integro-differential problems with constant coefficients, integral terms of the form (0.0.2), were already considered in \([26]\) and \([6]\). In \([49]\), a numerical scheme based on finite element methods in a convenient wavelet basis is proposed.

In this Thesis, we analyze finite difference schemes for the linear problem with constant coefficients. Two practical problems arise when discretizing the PIDE: the infinite domain and the non-local nature of the integral term. Actually, it is necessary to truncate the problem domain on one hand, and the integral domain on
the other. Moreover, once we have found a given domain, still we need to use some approximations of the solution in a larger computational domain. The common approach is to use some asymptotic representation formula for the solution. Here, we try a different and more general approach. First we notice that our original problem can be well approximated, by a pure differential problem with a suitable artificial diffusion. Then, we apply this remark to implement an effective numerical boundary condition, which reduces the numerical errors due to the truncation.

First, an explicit finite difference approximation is presented as direct application of the convergence result. Notice that this rigorous approach requires a diffusive CFL conditions, namely $\Delta t \sim \Delta x^2$. This condition is very heavy; the non-local nature of the integral term leads to high computational costs. In fact, at each space step of the discretization the integral term as to be computed. If $P$ is the number of grid points needed for this evaluation and $M$ is the number of grid points that defines the numerical domain, the computational cost, at each time step, will be $O(M^2)$ (notice that if a Fast Fourier Transform to compute the integral term is used, the cost decreases to $O(MP \log_2 P)$). Therefore, a diffusive CFL conditions is undesirable, since if $N$ is the number of time steps, then $N = O(M^2)$ and the global cost will be $O(M^3P^2)$.

To use smaller time step size, we need to involve some implicit mechanism in the scheme, but, due to the non-local nature of the integral term, a time implicit differencing scheme, unconditionally stable, is not practically feasible, and implicit schemes based on the FFT or wavelet basis fails to keep the monotonicity properties.

Then, to overcome these problems, we apply the Implicit-Explicit (IMEX) technique, which has been historically introduced for time dependent partial-differential equations that involve terms of different types, for instance for diffusion-convection equations [43, 17]. Typically, for these examples, an implicit scheme was used for the diffusion term and an explicit scheme was used for the convection term. Some schemes of this type were proposed and analyzed as far back as the late 1970’s. Instances of this methods have been successfully applied to the incompressible Navier-Stokes equations [41] and in environmental modeling studies [62]. We refer to [54, 52, 53] for recent developments.

Instead, here we use the IMEX schemes, by applying an explicit approximation both for the integral term and the convection term and an implicit approximation for the second order differential term. Then, we investigate the behavior of these approximations, studying their orders and their stability properties. The main point is that IMEX schemes are high order schemes under weaker stability condition, namely $\Delta t \sim \Delta x^\alpha$, $\alpha < 2$, in our example $\alpha = 4/3, 1$. The total computation cost is then reduced to $O(M^{1+\alpha}P^2)$ with $\alpha < 2$. Therefore, for $M \approx 10^3$, $P \approx 10^2$ and $\alpha = 1$, we reduce computations from $O(10^{13})$ to $O(10^9)$, namely by an order of $10^4$.

Numerical tests are presented to show these results and to find numerically the value of European options in the Merton’s model.
Plan of the Thesis

The thesis is organized as follows: Chapter 1, is devoted to the pricing problem in a jump-diffusion market. We first recall the concept of financial derivative and we give an overview of the pricing problem in the Black and Scholes model. In Section 1.2, we introduce the jump-diffusion modeling of the asset prices. The problem of the incompleteness of the market and the definition of the price are discussed and the connection between the stochastic pricing problem and the solution of an integro-differential equation is explained.

In Chapter 2 we give an introduction of Monte Carlo simulations for pricing in a jump-diffusion setting.

The purpose of Chapter 3 is to give an overview of analytical results for viscosity solutions for nonlinear equations. Starting from notions for the purely differential case, in Chapter 3.2 the definition of viscosity solution in the integral-differential setting is given. To deal with the option pricing problems, special attention is given to the well posedness of Cauchy problems in the class of exponential growth of the initial datum.

In Chapter 4, we present a background on finite differences approximations of solutions to linear partial differential equations. The main notions of consistency, stability, monotonicity and convergence of a numerical scheme are discussed and examined in detail through some examples. These concepts are then extended to nonlinear problems in Chapter 5. The general convergent result of monotone schemes for degenerate fully nonlinear second order equations is there exposed.

In Chapter 6, we provide the first analysis of the numerical approximation of viscosity solutions of integro-differential problems. First, we prove the general convergent result of monotone schemes for degenerate fully nonlinear second order integro-differential equations, then an explicit finite difference approximation is presented as direct application of this result. Practical difficulties arising from computing the numerical solution of integro-differential problems are there discussed.

Finally, in Chapter 7, high-order Implicit-Explicit approximations to linear integro-differential equations are examined. Moreover, some numerical tests show the effectiveness of these approximations. These scheme are then applied to find numerically the value of European options in the Merton’s model. In this framework, we compare these results with the numerical solution obtained by Monte Carlo simulations presented in Chapter 2.
Notations

We introduce some notations that shall be in use toward this Thesis.

∂t, partial derivatives with respect to time,

∂x, partial derivatives with respect to a direction xi,

Du = (∂x1u, ..., ∂xNu) gradient of u : ℝN → ℝ w.r.t. x ∈ ℝN,

D2u = (∂2xixj)i,j=1,...,N Hessian matrix of u w.r.t. x.

Let O be a subset of ℝN. We denote

C(O; X) = {f : O → X s.t. f is continuous},

and

Lp(O; X) = {f : O → X s.t. ∥f∥p < ∞},

where

∥f∥p = \begin{cases} \left( \int_O |f(x)|^p dx \right)^{1/p}, & \text{if } p \in [1, \infty), \\ \text{esssup}\{ |f(x)| \text{ s.t. } x \in O \} \|f\|_\infty, & \text{if } p = \infty, \end{cases}

with the convention that

∥f∥_p^p = ∥f∥_\infty \text{ when } p = \infty.

For X = ℝ we may write C(O) instead of C(O; X), as for Lp(O).

Moreover we set p* the conjugate exponent of p, namely

p* = p/(p-1) \quad \text{if } p \in (1, \infty),

p* = \infty \quad \text{if } p = 1,

p* = 1 \quad \text{if } p = \infty.

Let (Ω, ℱ, ℙ) be a probability space. We denote by X a random variable and by

𝔼(X) = \int_Ω X(ω) dℙ(ω) the expectation of X.

We denote

X = \{X_t : t ∈ [0, T]\} a stochastic process.

Namely X represents a function which associates to any time t a random variable X_t. Once an event ω ∈ Ω is selected, we define a path of the stochastic process X, the function

t ∈ [0, T], \ t → X_t(ω) ∈ ℝN.
If the probability space is endowed by a filtration \( \{ F_t; t \in [0,T] \} \), we denote by 

\[
E(X|F_t) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega),
\]

the conditional expectation of \( X \), conditioned to \( F_t \).

For any \( R > 0 \) we indicate 

\[
B(0,R) = \{ z \in \mathbb{R}^N : \| z \| < R \}.
\]

We set \( \mathcal{S}_N \) the set of symmetrical \( N \times N \) matrices with real coefficients, equipped with the usual order

\[
\mathcal{X} \leq \mathcal{Y} \quad \text{if} \quad \mathcal{X} \xi : \xi \leq \mathcal{Y} \xi : \xi \quad \text{for all} \quad \xi \in \mathbb{R}^N
\]

and the norm

\[
\| \mathcal{X} \| = \sup \left\{ |\mathcal{X} \xi : \xi| : \xi \in \mathbb{R}^N, \| \xi \| = 1 \right\}.
\]

Moreover, \( I_n = \text{diag}(1,\ldots,1) \) stands for the \( N \times N \) identity matrix and \( p \otimes q = (p_i q_j)_{i,j=1,\ldots,N} \) for the tensor product of two vectors of \( \mathbb{R}^N \).

Let \( A \subset \mathbb{R}^N \); we indicate with \( 1_A : \mathbb{R}^N \rightarrow \{0,1\} \) the characteristic function of the set \( A \).

We define rate of growth any nonnegative function \( g \in \mathcal{C}([0,\infty)) \) which is subadditive

\[
g(s + s') \leq g(s) + g(s').
\]

If moreover \( g(0) = 0 \), it is called a modulus of continuity.

We denote by \( \mathcal{C}^{2,1}(\mathbb{R}^N \times [0,T]) \) the set of functions which are twice continuously differentiable with respect to \( x \in \mathbb{R}^N \) and once with respect to \( t \in [0,T] \).

Finally, we define \( \text{USC} \) and by \( \text{LSC} \) the set of locally upper/lower bounded, upper/lower semicontinuous functions on \( \mathbb{R}^N \times [0,T] \).

Let us now consider an arbitrary subset \( \mathcal{O} \) of an Euclidean space \( \mathbb{R}^N \).

**Definition 0.0.1** A real function \( f \) is upper semicontinuous (resp., lower semicontinuous) at \( z \in \mathcal{O} \) if one of the following equivalent items is fulfilled:

1. \( u(z') \leq u(z) + o(\| z - z' \|) \) \quad (resp., \( \geq \))

as \( \mathcal{O} \ni z' \rightarrow z \). Here, \( o \) stands for a modulus of continuity.

2. \( \limsup_{\mathcal{O} \ni z' \rightarrow z} u(z') \leq u(z) \) \quad (resp., \( \liminf_{\mathcal{O} \ni z' \rightarrow z} u(z') \geq u(z) \)).

A trivial property of upper/lower semicontinuous functions is that they admit maximum (respectively, minimum) on every compact subset of \( \mathcal{O} \) in which they are upper (respectively, lower) bounded.
Definition 0.0.2 Given an arbitrary function $u : \mathcal{O} \to \mathbb{R}$, we define the upper semicontinuous envelope $u^*$ and the lower semicontinuous envelope $u_*$ of $u$ as point-wise infimum (respectively, supremum) of continuous functions staying above (respectively, below) $u$, namely

$$
\begin{align*}
&u^*(z) = \inf \left\{ v(z) : v \in \mathcal{C}(\mathcal{O}), v \geq u \text{ near } z \right\}, \\
u_*(z) = \sup \left\{ v(z) : v \in \mathcal{C}(\mathcal{O}), v \leq u \text{ near } z \right\}.
\end{align*}
$$

Upper/lower semicontinuous envelopes can be easily characterized as follows

$$
\begin{align*}
u^*(z) &= \limsup_{\mathcal{O} \ni z' \to z} u(z'), \\
u_*(z) &= \liminf_{\mathcal{O} \ni z' \to z} u(z').
\end{align*}
$$

To get well posedness for Cauchy problems, it is needed to fix a range of growth at infinity for solutions. So, for each rate of growth $g$, we set

$$
L^\infty_g(\mathbb{R}^N) = \left\{ f \in L^\infty_{loc}(\mathbb{R}^N) : \text{ there is } b \geq 0 \text{ s.t. } |f(x)| \leq bg(\|x\|) \text{ a.e. } \right\}
$$

$$
= \left\{ f \in L^\infty_{loc}(\mathbb{R}^N) : \text{ there is } b \geq 0 \text{ s.t. for all } R > 0 \left\| f; L^\infty_{[-R,R]^N} \right\| \leq bg(\|R\|) \right\};
$$

Viscosity solution theory requires to work with sub/supersolution, therefore one sided estimates are requested. We introduce subsets of $L^\infty_g$ that fits this request

$$
L^\infty_{g,+}(\mathbb{R}^N) = \left\{ f \in L^\infty_{loc}(\mathbb{R}^N) : \text{ there is } b \geq 0 \text{ s.t. } f(x) \leq bg(\|x\|) \text{ a.e. } \right\},
$$

$$
L^\infty_{g,-}(\mathbb{R}^N) = \left\{ f \in L^\infty_{loc}(\mathbb{R}^N) : \text{ there is } b \geq 0 \text{ s.t. } f(x) \geq -bg(\|x\|) \text{ a.e. } \right\}.
$$

If $g(r) = 1 + r$, the subscript “$g$” can be replaced with “lin”, standing for linear growth at infinity.

If $g_n(r) = 1 + r^n$, the subscript “$g_n$” stands for polynomial growth, with degree $n$, at infinity. In the following the subscript “pol” will denote the union over $n \geq 0$ of the correspondent sets subscripted by $1 + r^n$.

If $\tilde{g}_n(r) = e^{nr}$, the subscript “$\tilde{g}_n$” corresponds to an exponential growth, with weight $n$. In the following the subscript “exp” shall stand for the union over $n \geq 0$ of the correspondent sets subscripted by $e^{nr}$.

We define a numerical grid in $\mathbb{R}^N \times (0,T)$ using the following notations:

- $h = (h_1, ..., h_N)$ is the spatial grid size;
- $k$ is the time grid size;
- $(x_m, t_n) = (mh, nk)$ for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ are the grid points;

For a function $v$ defined on the grid we write $v^n_m = v(x_m, t_n)$. We also use the notation $u^n_m$ for $u(x_m, t_n)$ when $u$ is a continuous function on the $(x,t)$ plane and $\tilde{v}$ for the vector of $v$ values, $(v^n_m)_{m,n}$ for $m$ and $n$ varying on a subset of $\mathbb{Z}$ and $\mathbb{N}$ respectively.
Chapter 1

Pricing in a jump-diffusion market

In this chapter we shall give some notion on pricing in a jump-diffusion model. For a complete description of the basic financial concepts in a mathematical setting we refer to the book of Lamberton and Lapeyre [45] and to [39] for rigorous notions on stochastic calculus.

We shall consider a financial market where two assets \((B, S)\) are traded continuously up to some fixed time horizon \(T\). The market shall be defined by a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a right continuous filtration \(\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}\) with \(\mathcal{F}_T = \mathcal{F}\).

The basic objects of study will be the financial derivatives. These are asset which in some way are defined in terms of the underlying asset \(S\). Typical examples are options, forwards, futures, bonds, interest rate swaps, caps and floors. The mathematical formalization of a derivative may be given as

**Definition 1.0.3** A contingent claim with maturity \(T\) is an arbitrary \(\mathcal{F}_T\)-measurable random variable \(X\).

The interpretation of this definition, shall be that the contingent claim is a contract which specifies that the stochastic amount \(X\) of money is to be payed out to the holder of the contract at time \(T\).

**Example 1.0.4** The most popular (and historically the oldest) example of a contingent claim is that of a European call option on \(S\) with strike price \(K\) and maturity \(T\). This \(T\)-contract is defined by the random amount

\[
X = \max [S_T - K, 0], \tag{1.0.1}
\]

and gives to the owner the right, but not the obligation, to by one share of the stock \(S\) at the prespecified price \(K\), at time \(T\).

On the other hand a European put option on \(S\) with strike price \(K\) and maturity \(T\) is the \(T\)-contract defined by

\[
X = \max [K - S_T, 0], \tag{1.0.2}
\]
that gives to the owner the right, but not the obligation, to sell one share of the stock at the prespecified price $K$, at time $T$.

We shall investigate the problem of pricing: what is the “fair” price for a given contingent claim $X$?

Our attention will be mostly focused on the relations between second order parabolic differential equations and stochastic processes. This relation explains why numerical approximations of option prices can be obtained either by using Monte Carlo methods or by solving partial differential equations.

### 1.1 The Black and Scholes model

Let us consider a financial market defined by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$, and let $W = (W_t)_{0 \leq t \leq T}$ be a standard Brownian motion adapted to this filtration. The Black and Scholes market model is defined as follows

\[
\begin{align*}
    dB_t &= r B_t dt, \\
    dS_t &= \mu S_t dt + \sigma S_t dW_t.
\end{align*}
\]

The short rate of interest $r$, the local mean rate return $\mu$ and the local volatility $\sigma$ are assumed to be known deterministic constants. It is well known that the Black and Scholes market is without arbitrage opportunities and it is complete. Then, the following classical result for the price of a simple contingent claim holds (see for instance [45, Chapter 5]).

**Proposition 1.1.1** Consider the Black and Scholes model and a square integrable $T$-claim $X$. Then the price process $\Pi(X,t)$ is given by

\[
\Pi(X,t) = e^{-r(T-t)} \mathbb{E}^Q[X|\mathcal{F}_t],
\]

where the $\mathcal{Q}$-dynamic of $S$ is given by

\[
dS_t = rS_t dt + \sigma S_t dW_t.
\]

If furthermore $X$ is of the form $X = \phi(S_T)$, then the price process is also given by $\Pi(X,t) = V(S_t, t)$, where the function $V \in C^{1,2}([0,T] \times \mathbb{R})$ solves the Black and Scholes equation

\[
\begin{align*}
    \frac{\partial V}{\partial t}(S,t) + rS \frac{\partial V}{\partial S}(S,t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) - rV(S,t) &= 0, \\
    V(S,T) &= \phi(S).
\end{align*}
\]
1.2 The jump-diffusion model

Most of the literature in finance in particular for pricing and hedging of contingent claims, is based on the assumption that the prices of the underlying assets follow a diffusion-type process as in the Black and Scholes model. Various empirical studies (see for instance [18]) show that such models are inadequate, both in relation to their descriptive power as well as for the mispricing that they might induce. Here we concentrate on the fact that returns of asset prices may exhibit a jumping behavior. The main reference for this section is [58].

Let us consider a financial market where two assets \((B,S)\) are traded continuously up to some fixed time horizon \(T\). The market is defined by a probability space \((\Omega, \mathcal{F}, P)\). \(\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}\) is a complete right continuous increasing filtration with \(\mathcal{F}_T = \mathcal{F}\). The process \(W = (W_t)_{0 \leq t \leq T}\) is a standard Brownian motion adapted to this filtration. On this probability space is also defined an integer-valued random measure \(v(dt, dz)\) associated to the marked point process \((N_t, (U_n))_{n \in \mathbb{N}}\) where:

- \(N_t\) is a counting process corresponding to the random time points \((T_n)_{n \in \mathbb{N}}\);
- \((U_n)_{n \in \mathbb{N}}\) is a sequence of random variables with values in a subset of \(\mathbb{R}\).

By abuse of language, \(v\) and \((N_t, (U_n))_{n \in \mathbb{N}}\) may be identified by the formula

\[
v([0, t] \times E) = \sum_{n \geq 1} 1_{T_n \leq t} 1_E(U_n),
\]

for all \(0 \leq t \leq T\) and \(E \in \mathcal{B}\) (\(\mathcal{B}\) is the Borel \(\sigma\)-field on \(U\)).

The Brownian \(W\) and the marked process \(v\) are assumed to be independent and \(\mathcal{F}\) is assumed to be the canonical filtration generated by \((W, v)\).

We suppose that there is a finite number of jumps during any finite time interval and we shall assume that:

(i) \(v\) is a compound Poisson process with constant intensity \(\lambda\);

(ii) \(v(dz)\) is the probability measure under \(P\) of the i.i.d. random variables \(U_1, ..., U_n\), independent of \(N_t\);

(iii) the diffusion volatility \(\sigma\) is constant and the jump sizes \(\gamma(z)\) does not depend on \(t \in [0, T]\) and it is such that

\[
\int_{\mathbb{R}} \gamma^2(z) \nu(dz) < +\infty; \tag{1.2.1}
\]

(iv) the interest rate \(r\) is constant.
By defining the compensated jump martingale of \( v \)
\[ \tilde{v}(dt, dz) = v(dt, dz) - \lambda \nu(dz), \]
we have that \( (\tilde{v}([0, t] \times E))_{0 \leq t \leq T} \) is a \((\mathbb{P}, \mathcal{F}_t)\) martingale (see for instance [58]).

Then, we consider the following jump-diffusion market
\[
\begin{aligned}
B_t &= e^{rt}, \\
\frac{dS_t}{S_t^{-}} &= \mu_t dt + \sigma dW_t + \int_{\mathbb{R}} \gamma(z) \tilde{v}(dt, dz).
\end{aligned}
\]

(1.2.2)

The interpretation that should be given to the various parameters is as follows: \( \mu \) is the drift of the asset under \( \mathbb{P} \), and \( \sigma \) the volatility of the Brownian part. The rate of jumps in the price process is given by \( \lambda \); \( (T_n) \) are the random time points of the jumps and a jump at time \( T_n \) with amplitude \( U_n \) corresponds to a proportional jump in the asset price process of size \( \gamma(U_n) \) with \( U_n \) distributed according to the probability law \( \nu(dz) \). In order to ensure limited liability \( (S_t \geq 0) \), we assume that \( \gamma \geq -1 \).

**Example 1.2.1** [Merton’s model] The prototype of jump-diffusion market has been first proposed by Merton [50]. The underlying process dynamic is given by the following equation:
\[
\frac{dS_t}{S_t} = (\mu - \lambda \mathbb{E}[\eta - 1]) dt + \sigma dW_t + (\eta - 1) dN_t,
\]

(1.2.3)

where \( W_t \) is a standard Brownian motion, \( N_t \) a Poisson counting process of intensity \( \lambda \), that is:
\[
dN_t = \begin{cases} 
0 & \text{with probability } 1 - \lambda dt \\
1 & \text{with probability } \lambda dt,
\end{cases}
\]

\( dN_t \) and \( dW_t \) are assumed to be independent. If \( \lambda = 0 \) (and therefore, \( dN_t \equiv 0 \)), then the return dynamic would be identical to those presented by Black & Scholes (1.1.1). Moreover, the relative change \( \eta - 1 \) in the stock price has been assumed to be a random variable log-normal distributed, with probability density:
\[
\tilde{\Gamma}_\delta(x) = \frac{1}{x \delta \sqrt{2\pi}} \exp \left( - \frac{1}{2} \left( \frac{\log x}{\delta} \right)^2 \right).
\]

(1.2.4)

It is possible to generalize these models and to extend the analysis to include models where the jumps (or equivalently the right-hand-side of (1.2.2)) form a Lévy process, this is the approach taken by Chan [19].
1.2.1 Incompleteness and risk neutral measures

Let $X$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that defines an option written on the jump-diffusion market (1.2.2). To price options, we use the modern theory of derivatives valuation based on the arbitrage principle and the martingale representation.

**Definition 1.2.2** An equivalent martingale measure is a probability measure $\mathbb{Q}$ on $\mathcal{F}$ equivalent to $\mathbb{P}$ such that the discounted traded asset price $S_t/B_t$ is a $\mathbb{Q}$ martingale.

It is known (see for instance [45]) that the existence of an equivalent martingale measure, implies the absence of arbitrage opportunities in the market $(B, S)$. The inverse is not always true in continuous time.

The existence of an equivalent martingale measure $\mathbb{Q}$ in a jump-diffusion market (1.2.2), is ensured by the following result.

**Proposition 1.2.3 ([19, 55, 12])** The equivalent martingale measures are characterized by their Radon-Nikodym density with respect to $\mathbb{P}$:

$$
\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp \left( -\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right) \\
\times \exp \left( \int_0^t \int_{\mathbb{R}} \ln(q_s(z)) v(ds, dz) - \int_0^t \int_{\mathbb{R}} (q_s(z) - 1) \lambda \nu(dz) ds \right),
$$

where $\theta$ and $q$ are two predictable processes such that

$$
\mu_t - r_t = \theta_t \sigma_t + \lambda \int_{\mathbb{R}} \gamma(z)(1 - q_t(z))dz,
$$

(1.2.5)

together with the conditions

$$
qu > 0 \text{ and } \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \right] = 1.
$$

**Remark 1.2.4** The excess rate of return can be divided into two parts: the premium for the continuous part of the return and the premium for the jump part. The process $\theta$ may be defined as the market price of diffusion risk and $q$ as the market price of jump risk, which are not uniquely determined by the relation (1.2.5): the market $(B, S)$ is incomplete (an intuitive idea of what is the meaning of completeness of a market can be found in [13] as the so called Meta Theorem). The equivalent martingale measures are parameterized by $q$ and we shall note by $\mathbb{Q}^q$ the associated equivalent martingale measure.

**Remark 1.2.5** The Merton model (1.2.3), corresponds to the case $q = 1$. In [50], options were valued by arbitrage arguments as is the Black and Scholes one [14], and by assuming that the jump risk premium is not priced.
From Girsanov’s theorem (see for instance [45, Chapter 4] and [58]), we have that $W^q_t = W_t + \int_0^t \theta^q_s ds$ is a $Q^q$ Brownian motion and $\nu$ is a $(Q^q, F_t)$ marked point process with a predictable intensity $\lambda^q_t(z)\nu(dz)$. Moreover, we have that the risky asset in (1.2.2), satisfies the following stochastic equation under $Q^q$,

$$
\frac{dS_t}{S_t} = rd t + \sigma dW_t^q + \int_{\mathbb{R}} \gamma(z)(\nu(dt,dz) - \lambda^q_t(z)\nu(dz)dt).
$$

(1.2.6)

Since the jump-diffusion market (1.2.2) is incomplete (Remark 1.2.4), the derivation of the fair price of derivatives becomes somewhat more involved. Three different procedures are commonly used to price derivatives in incomplete market.

1. The first one consists in the completion of the market adding as many new assets as the sources of uncertainty, [36, 37, 38, 60]. Notice that, in this case the dimension of the problem increases and the possible correlation between assets leads to a degenerate partial integro-differential, as we shall show in Example 1.2.10.

2. This second approach and the next one choose to deal with the pure primary market. There is not in general a unique arbitrage price for a given contingent claim, but according to [32], an infinity of viable prices corresponding to each equivalent martingale measure $Q^q$. Then, the procedure relies on picking out a “fair price” for derivatives by selecting a feasible price for risk. Over the years several approaches for selecting an equivalent martingale measure have been suggested in literature (see for instance [59, 20]). For example, one may opt for minimizing the risk (in this case it is needed to calculate the so called minimal martingale measure) or for the super-hedging, that is based on the idea of “protecting from the worst” (in this case, it is needed to find the maximal martingale measure).

3. The last procedure consists in choosing the point of view of economy and in selecting an unique price for derivatives by imposing the maximization of utility at equilibrium. This strategy leads to the solution of some stochastic control problems, which may brought back to the one of a deterministic partial differential equation by means of the dynamic programming principle and of the theory of viscosity solutions (see for instance [28]).

We focus now on the second approach; once a strategy to select an equivalent martingale measure have been chosen, the arbitrage price with respect to this measure, of an European option $X = \phi(S_T)$ expiring at time $T$ can be defined by

$$
\Pi^q(\phi, t) = \mathbb{E}^{Q^q}[e^{-r(T-t)}\phi(S_T)|F_t].
$$

(1.2.7)

In the next section, we shall characterize this formula by the solution of an integro-differential Cauchy problem.
1.2.2 The linear integro-differential problem

Starting from the discussion given in the previous section, we rewrite the dynamic of \( S \), given in (1.2.2), in the more general form

\[
dS_t = b(S_t^{-}, t)dt + \sigma(S_t^{-}, t)dW_t + \int_{\mathbb{R}} \gamma(S_t^{-}, t, z)\tilde{v}(dt, dz).
\]

(1.2.8)

For a given smooth function \( v(y, t) \), by the generalized Ito calculus (see for instance [58]), we shall consider the linear integro-differential operator associated to the jump-diffusion process \( S_t \) solution of (1.2.8),

\[
\mathcal{L}v = \frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2(t, y) \frac{\partial^2 v}{\partial y^2} + b(y, t) \frac{\partial v}{\partial y} + \lambda \int_{\mathbb{R}} \left( v(y(1 + \gamma(y, t, z)), t) - v(y, t) - \gamma(y, t, z) \frac{\partial v}{\partial y} \right) \nu(dz) - c(t, y)v.
\]

(1.2.9)

We shall look for a smooth solution \( C^{1,2} \) (\( C^1 \) in \( t \) and \( C^2 \) in \( y \)), of the Cauchy problem

\[
\begin{cases}
\mathcal{L}v = f(y, t) & \forall (y, t) \in \mathbb{R} \times [0, T), \\
v(y, T) = \phi(y) & \forall y \in \mathbb{R}.
\end{cases}
\]

(1.2.10)

We assume that the function \( b, \sigma, \gamma, c, f, \phi \) are all continuous and:

\( \text{(L1)} \) \( \sigma(y, t) > 0 \) for all \( (y, t) \in \mathbb{R} \times [0, T] \).

\( \text{(L2)} \) \( b \) and \( \sigma \) are bounded and locally Lipschitz in \( (y, t) \).

\( \text{(L3)} \) There exists a function \( \rho : \mathbb{R} \to \mathbb{R}^+ \) with \( \int_{\mathbb{R}} \rho^2(z)\nu(dz) < +\infty \) such that

\[
|\gamma(y, t, z)| \leq \rho(z) & \forall (y, t) \in \mathbb{R} \times [0, T], \\
|\gamma(y, t, z) - \gamma(x, t, z)| \leq \rho(z)|y - x| & \forall (y, x, t) \in \mathbb{R} \times \mathbb{R} \times [0, T], \\
(y, t) \to \int_{\mathbb{R}} \gamma(y, t, z)\nu(dz) \text{ is locally Lipschitz in } (y, t).
\]

(1.2.11)

\( \text{(L4)} \) There exists \( K > 0 \) such that for all \( t \in [0, T], x, y \in \mathbb{R} \):

\[
|b(y, t) - b(x, t)| + |\sigma(y, t) - \sigma(x, t)| \leq K|y - x|.
\]

(1.2.12)

\( \text{(L5)} \) There exists \( K > 0 \) such that for all \( t, s \in [0, T], x, y \in \mathbb{R} \):

\[
|f(y, t) - f(x, s)| + |\phi(y) - \phi(x)| \leq K||s - t| + |y - x||.
\]
Pricing in a jump-diffusion market

(L6) $c$ is bounded and locally H"older continuous in $(y,t)$.

- $\nu$ is a bounded measure.

Then, the following result holds.

**Proposition 1.2.6** [56, Proposition 5.3] Under the assumptions (L1)-(L6), there exists a unique solution $v \in C^{1,2}(\mathbb{R} \times [0,T]) \cap C^0(\mathbb{R} \times [0,T])$ of the Cauchy problem (1.2.10) that satisfies

$$|v(y,t)| \leq C(1 + |y|^p) \quad \forall (y,t) \in [0,T] \times \mathbb{R}$$

for some constant $C$ and $p \in [0,2)$. This solution is given by

$$v(y,t) = \mathbb{E}_{yt} \left[ \int_t^T e^{-\int_t^T c(u,S_u)du} f(S_\tau,\tau)d\tau + e^{-\int_t^T c(u,S_u)du} \phi(S_T) \right],$$

where $\mathbb{E}_{yt}$ is the conditional expectation under $\mathbb{P}$ given that $S_t = y$.

Coming back to the option pricing problem, we now consider again an European option $X = \phi(S_T)$ expiring at time $T$ and we shall assume that the market price of jump risk does not depend on $t$, $q_t(z) = q(z)$, that $q$ is square integrable with respect $\nu$ and that $\phi \in C^0$.

Let $S_t(y)$ be the solution of (1.2.6) with initial value $y$, the Proposition 1.2.6 allows us to relate the function

$$V(y,t) = \mathbb{E}^{Q_\theta}[e^{-\int_t^T \phi(S_T)}],$$

which defines the arbitrage option price under the equivalent martingale measure $Q_\theta$, to the solution of a parabolic integro-differential Cauchy problem.

Notice that $\sigma(S,t) = \sigma S$ in (1.2.6) and the value $S = 0$ shall be a degeneration point for the equation (see condition (L1)). Therefore, it turns out to be easy to express the option price in term of the logarithmic value of the stock price: we therefore use the logarithmic change of variable $x = \log(y)$ and the unknown function $u(x,t) = V(e^x,t)$. Then, by Proposition 1.2.6, the function $u(x,t)$ is the smooth solution of the Cauchy problem

$$\left\{ \begin{array}{ll}
\mathcal{L}^\theta u = 0 & \forall (x,t) \in \mathbb{R} \times [0,T), \\
u(x,T) = \phi(e^x) & \forall x \in \mathbb{R},
\end{array} \right. \quad (1.2.12)$$

where

$$\mathcal{L}^\theta u = \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{1}{2} \sigma^2) \frac{\partial u}{\partial x} - ru + \lambda \int_\mathbb{R} \left[ u(e^x(1 + \gamma(z)),t) - u(x,t) - \gamma(z) \frac{\partial u}{\partial x} \right] q(z) \nu(dz). \quad (1.2.13)$$
**Example 1.2.7** Let us assume $q \equiv 1$ and $\gamma(\eta) = \eta - 1$. The integro-differential pricing equation (1.2.12), becomes

$$
-\frac{\partial u}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \left( r - \frac{1}{2} \sigma^2 - \lambda \mathbb{E}[\eta - 1] \right) \frac{\partial u}{\partial x} - ru
+ \lambda \int_{\mathbb{R}} [u(e^\eta, t) - u(x, t)] \nu(\eta) d\eta,
$$

(1.2.14)

where $\mathbb{E}[\eta - 1] = \int_{\mathbb{R}} (\eta - 1) \nu(\eta)$. Choosing as in the Merton model (Example 1.2.1), the relative jumps amplitude $\eta - 1$ to be log-normal distributed and using the change of variable $z = \log \eta$, the integral term in (1.2.14) is

$$
\mathcal{I}u = \lambda \int_{\mathbb{R}} [u(x + z, t) - u(x, t)] \Gamma_\delta(z) dz,
$$

where

$$
\Gamma_\delta(z) dz = \frac{1}{\sqrt{2\pi\delta}} \exp\left( -\frac{z^2}{2\delta^2} \right) dz.
$$

(1.2.15)

---

**1.2.3 Nonlinear models**

In this section we shall give some examples of models that involves the solution of possibly degenerate nonlinear parabolic problems. These models may not be approached by the classical solution theory (as given in the previous section). In Chapter 3 we shall give an overview of viscosity solutions, that are weak enough to ensure existence, still assuring the uniqueness of the solution of such problems. In Chapter 6 effective numerical schemes will be introduced to provide well posed numerical solutions. Moreover, some of the examples we are going to introduce, will be used as numerical tests.

As a relevant example we mention the *large investor* models. Nowadays the classical assumption of *small investor* (whose strategy may not influence the market prices) appears to be too restrictive. When dealing with a market where the economic parameter (drift, volatility and interest rate) are allowed to depends on the traders’strategies, nonlinear effects arise, so that the price of derivatives is described by a nonlinear stochastic differential equation.

Following [5, 3, 63], let

$$
\xi(V, DV, JV) = V - S\theta_0 DV - \phi_0 JV
$$

denote the amount of money invested in stocks by the agent, obtained choosing a proper replicating portfolio, for some given constants $\theta_0$ and $\phi_0$, see [5]; then the interest rate $r$ is influenced by the agents by means of $\xi$.  

---

**The jump-diffusion model**
Example 1.2.8 [Different interest rates for borrowing or lending] Here we assume that the interest rate is given by

\[ r(S,t,\xi) = \begin{cases} R(S,t) & \text{if } \xi \leq 0, \\ \rho(S,t) & \text{if } \xi > 0, \end{cases} \]

where \( R \) and \( \rho \) are continuous functions and \( R(S,t) > \rho(S,t) \).

Example 1.2.9 [Large institutional investor] In this second case, the interest rate decreases when too many wealth is invested in bonds, according to the law \( r(S,t,\xi) = R(S,t)f(S-\xi) \), where \( f \in C(\mathbb{R}) \) is positive, \( f(\xi) = 1 \) as \( \xi \leq \xi_0 \) and decreasing as \( \xi > \xi_0 \), but \( f(\xi) \) non decreasing.

In this setting the price function \( V \) must solve the following quasi-linear final value problem:

\[
\begin{cases}
\partial_\tau V + L_\mathcal{J} V = H(S,\tau,V,\mathcal{J}V,DV), \\
V(S,T) = G(S),
\end{cases}
\]  

(1.2.16)

where

\[
L_\mathcal{J} V = \frac{1}{2} S^2 \sigma^2 D^2 V + S[\mu + \sigma \theta] D V - \phi \mathcal{J} V,
\]

\( \mathcal{J} V \) is the integral operator

\[
\mathcal{J} V(S,t) = \int_\mathbb{R} [V(Se^z,t) - V(S,t)] \nu(dz).
\]

and \( H \) is a nonlinear first order operator:

\[
H(S,\tau,V,\mathcal{J}V,DV) = r(S,\tau,\xi) \cdot \xi.
\]  

(1.2.17)

Obviously, if all the parameters of the model \( r, \mu, \sigma \) are deterministic function of \((S,\tau)\), the problem is linear and we obtain again the so called small investor economy described in the previous sections.

Example 1.2.10 [Completion of the market in the large investor model] It is proved that a jump-diffusion market is incomplete because of the arbitrage opportunities arising at the jump time (see [13]). Several approaches may be used to overcome the difficulty of incompleteness. One of them is the completion of the market adding as many new assets as the sources of uncertainty, [36, 37, 38, 60]. A standard approach is to add a call which parameters are taken directly from the market, and not by Ito rule. Therefore we can suppose that our market is described by

\[
\begin{cases}
dS_t^0 = S_t^0 r(X,t,\xi) dt, \\
dX = X_0 dt + X \beta dW_t + X(\gamma - 1) dN_t,
\end{cases}
\]
where $X_t = \text{diag}(S_t, C_t)$ and the vectors of expectation $\alpha$, volatility $\beta$, and jump $\gamma$ are:

$$
\alpha = \begin{pmatrix}
\mu - \lambda k \\
\mu_c
\end{pmatrix}, \quad
\beta = \begin{pmatrix}
\sigma \\
\sigma_c
\end{pmatrix}, \quad
\gamma = \begin{pmatrix}
\eta \\
\eta_c
\end{pmatrix}.
$$

We shall suppose, the jump amplitude $\gamma$ is now log-normally distributed with density

$$
\hat{\Gamma}(\gamma) = \hat{\Gamma}(\eta) \cdot \hat{\Gamma}(\eta_c).
$$

In this frame set the pricing equation is the extension of (1.2.16) to the multidimensional case:

$$
\begin{align*}
\partial_t V + L_{\mathcal{J}} V &= H(S, t, V, \mathcal{J}V, D^2 V), \\
V(X, T) &= G(X).
\end{align*}
$$

Here, the operator

$$
L_{\mathcal{J}} V = \frac{1}{2} \text{tr}[(X\beta)(X\beta)^T D^2 V] + X[\alpha + \beta \theta] \cdot D^2 V - \phi \cdot \mathcal{J}V,
$$

is linearly degenerate elliptic. Moreover $H$ has the same form as (1.2.17), with $X$ playing the role of $S$, and $\mathcal{J}V$ as previously. We can note that in this case the diffusion matrix is degenerate, since

$$
\text{rk}((X\beta)(X\beta)^T) < 2.
$$

If we apply a change of variable in order to have diffusion only in one direction:

$$
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
\cos \vartheta & \sin \vartheta \\
-\sin \vartheta & \cos \vartheta
\end{pmatrix} \cdot \begin{pmatrix}
\log S \\
\log C
\end{pmatrix},
$$

with $\vartheta = \arctg \frac{\sigma_c}{\sigma}$, and proper coefficients $A, B, C, D$ we obtain:

$$
\begin{align*}
\partial_t u + \frac{1}{2} (\sigma^2 + \sigma_c^2) \partial_{xx}^2 u &+ A \partial_x u + B \partial_y u - \phi I u = \\
&= r \left( u + C \partial_x u + D \partial_y u - \phi_0 I u \right) \times \\
&\times \left( u + C \partial_x u + D \partial_y u - \phi_0 I u \right),
\end{align*}
$$

where:

$$
I_u = \lambda \left( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x + \xi, y + \zeta, t) \exp \left( -\frac{\xi^2 + \zeta^2}{2\delta^2} \right) \cdot \exp \left( -\frac{(\delta^2 - \rho^2)(\xi \sin \theta + \zeta \cos \theta)^2}{2\delta^2 \rho^2} \right) d\xi d\zeta - u(x, y, t) \right).
$$

\[\square\]
Chapter 2

Monte Carlo simulations

As we have shown in the previous chapter the price of an European derivative which guarantees the payoff \( \phi(S_T) \) at time \( T \), may be calculate taking the conditional expectation of its discounted value,

\[
V_t = \mathbb{E}\left[ e^{-r(T-t)} \phi(S_T) | \mathcal{F}_t \right],
\]

where \( r \) is the riskless interest rate, \( S_t \) is the price of the underlying asset at time \( t \) and \( \mathbb{E} \) denotes the expectation with respect to the risk neutral probability.

In this chapter we give an introduction to Monte Carlo methods. This method is used to evaluate an integral as an expected value. We will consider the following problem

\[
I = \mathbb{E}_\mu[\psi(X)] = \int_D \psi(x) d\mu(x)
\]

where \( \psi \) is some function on \( D \subseteq \mathbb{R} \) and \( X \) is a random variable with law \( \mu \).

The first section is devoted to Monte Carlo Simulations, principle of estimation, variance reduction techniques and efficiency of the simulation. The next section deals with the simulation of random variables. We present some algorithms, especially for Gaussian and Poisson variables. Finally we describe how to simulate diffusion processes and jump-diffusion processes. Such algorithms are necessary to price financial options with simulation methods.

For a more detailed review on these methods, we refer for instance to [57], [15], [31] or [21]. A reference for applications in Finance is [46].

2.1 Principle of the Monte Carlo Simulations

In this section we discuss a mathematical model for “repeated, independent experiments”, that is the base of the Monte Carlo simulation. The idea is this: suppose we are considering a probability space and a real valued random variable \( X \) on it, which records the outcome of some sort of random experiment. We can model repetitions
of this experiment by introducing a sequence of random variables \( X_1, \ldots, X_n, \ldots \) each of which “has the same probability information as \( X \)".

**Definition 2.1.1** A sequence \( X_1, \ldots, X_n, \ldots \) of random variables is called identically distributed if
\[
F_{X_1}(x) = F_{X_2}(x) = \ldots = F_{X_n}(x) = \ldots, \quad \text{for all } x,
\]
where \( F_X(\cdot) \) is the distribution function of \( X \).

If we additionally assume that the random variables \( X_1, \ldots, X_n, \ldots \) are independent, we can regard this sequence as a model for repeated and independent runs of the experiment.

We first give the theorem that shows that with probability one, we can deduce the common expected values of the random variables.

**Theorem 2.1.2 (Strong Law of Large Numbers)** Let \( X_1, \ldots, X_n \) be a sequence of independent, identically distributed, integrable random variables defined on the same probability space, such that for \( i = 1, \ldots, n \),
\[
m = \mathbb{E}[X_i].
\]
Then
\[
P\left( \lim_{n \to \infty} \frac{X_1 + \ldots + X_n}{n} = m \right) = 1.
\]
The Strong Law of Large Numbers says that for almost every sample point \( \omega \in \Omega \),
\[
\frac{X_1(\omega) + \ldots + X_n(\omega)}{n} \to m \quad \text{as } n \to \infty.
\]
Therefore, if \( X_1, \ldots, X_n \) is a sequence of random variables each of which “has the same probability information as \( X \)” and \( \mathbb{E}[X] < +\infty \),
\[
\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mathbb{E}[X].
\]
We have then found an unbiased estimation of \( I \) defined in (2.0.2): for \( n \) trial, if
\[
\theta_n = \frac{1}{n} \sum_{i=1}^{n} \psi(X_i), \quad (2.1.1)
\]
with \( X_i \) i.i.d. random variables to \( \mu \), we have
\[
\theta_n \xrightarrow{a.s.} I.
\]

We turn now to the Central Limit Theorem, which estimates the error we can expect in the approximation (2.1.1).
Theorem 2.1.3 (Central Limit Theorem) Let $X_1, ..., X_n$ be a sequence of independent, identically distributed, real-valued random variables with, for $i = 1, ..., n$

$$\mathbb{E}[X_i] = m, \ Var[X_i] = \sigma^2 > 0.$$ 

Set

$$S_n = X_1 + ... + X_n.$$ 

Then for all $-\infty < a < b < +\infty$

$$\lim_{n \to \infty} \mathbb{P}\left(a \leq \frac{S_n - nm}{\sigma \sqrt{n}} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx. \quad (2.1.2)$$

Hence the Central Limit Theorem says that the sum $S_n$, properly renormalized, have a distribution which tends to the Gaussian $\mathcal{N}(0, 1)$ as $n \to \infty$.

Therefore, if $\sigma^2 = Var[\psi(X)] < +\infty$, the error of the approximation (2.1.1) has the following property

$$\frac{\sqrt{n}}{\sigma} \left(\frac{1}{n} \sum_{i=1}^{n} \psi(X_i) - \mathbb{E}[\psi(X)]\right) \xrightarrow{L} \mathcal{N}(0, 1).$$

The variance of the estimator is then given as

$$\sigma_n^2 = \frac{\sigma^2}{n}.$$ 

It decreases to zero when $n \to \infty$. It means that the greater $n$ is, the more accurate the estimator is. The speed of convergence of $\theta_n$ to $I$ is $\sigma/\sqrt{n}$.

Moreover, from the Central Limit Theorem, we can define a confidence interval for the threshold $1 - 2\alpha$. It is such that $\mathbb{P}(A < I < B) = 1 - 2\alpha$ and it is built as follows:

$$[\theta_n - z_\alpha \sigma_n; \theta_n + z_\alpha \sigma_n]$$

where $z_\alpha = \Phi^{-1}(1 - \alpha)$ and $\Phi^{-1}$ is the inverse cumulative distribution function of the standard Gaussian law. For instance, if the threshold is chosen to 95% then $\alpha = 2.5\%$ and $z_\alpha \approx 1.96$.

We briefly summarize some advantages and disadvantages of the Standard Monte Carlo method.

- **Advantages:** This method does not require regularity or differentiability properties for the function $\psi$. Thus we can implement it very easily if we are able to generate the variable $X$ according to $\mu$. Moreover, error on the estimate can be controlled by the Central Limit Theorem.

- **Disadvantages:** We have to realize a lot of simulations to obtain an accurate estimator. Therefore computing time can be very high.
2.2 Simulation of random variables

To realize a Monte Carlo simulation, we need a sample of $n$ random variables $X_i$ i.i.d. according to the same density, as we have described in the previous section. In this section we present some general methods to obtain this random sample to numerically solve financial model. First, we recall that simulation of most of the random variables requires uniform variables over $[0, 1]$, that are random numbers. Most of computer programming languages have a random function that return pseudo-random variables on $[0,1]$ (for example the C function $rand()$).

The density distributions we need to simulate, to solve financial model are essentially the Gaussian distribution, the exponential distribution and the Poisson one. We shall give briefly some methods to simulate them.

Simulation of Gaussian standard variables

The standard method to simulate Gaussian random variables, is based on the property that if $(u,v)$ are two independent random variables uniformly distributed on $[0,1]$, then $x$ and $y$ defined by:

$$
x = \sqrt{-2 \log u} \sin(2\pi v) \\
y = \sqrt{-2 \log u} \cos(2\pi v)
$$

are $\mathcal{N}(0,1)$. Therefore, to simulate a Gaussian random variable $X$, such that $\mathbb{E}[X] = m$ and $\text{Var}[X] = \sigma$, it is sufficient to define $X = m + \sigma x$ or $X = m + \sigma y$.

Simulation of exponential random variables

We recall that $X$ is an exponential random variable with parameter $\lambda$ if it has law

$$1_{x \geq 0} \lambda e^{-\lambda x} dx.$$

Furthermore, if $u$ is uniformly distributed on $[0,1]$, the random variable $X = \log (u)/\lambda$ is exponentially distributed.

Simulation of Poisson random variables

A Poisson random variable is such that:

$$\mathbb{P}(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}, \text{ if } n \geq 0.$$

The method we describe is based on the following fact: if $(T_i)_{i \geq 1}$ is a sequence of exponential random variables with parameter $\lambda$, then the counting process $N_t = \sum_{n \geq 1} 1_{(T_1+\ldots+T_{n+1}) \leq t < T_1+\ldots+T_{n+1}}$ has Poisson law with parameter $\lambda t$. Moreover, as we have previously shown we can always write the exponential variables $T_i$ as
An idea on simulations of Diffusion and Jump-Diffusion Processes

$(-\log (U_i)/\lambda)$, where $(U_i)_{i\geq 1}$ are independent random variables uniformly distributed on $[0, 1]$. The Poisson random variable $X = N_1$ can then be written as

$$N_1 = \sum_{n\geq 1} n 1\{U_1 U_2 \cdots U_{n+1} \leq e^{-\lambda} \leq U_1 U_2 \cdots U_n\}.$$ 

Therefore the algorithm becomes:

1. Set $a = \exp(-\lambda)$, $n = 0$, $u = \text{Random}();$
2. While $(u > a)$ set $u = u \cdot \text{Random}(), n = n + 1;$
3. Deliver $n$.

2.3 An idea on simulations of Diffusion and Jump-Diffusion Processes

This section is a brief introduction to the simulation of stochastic processes, it is especially devoted to simulation of diffusion process and it contains an algorithm to simulate jump-diffusion processes.

2.3.1 Simulation of Brownian Motion

Let us first recall the definition of the Wiener process (or Brownian motion).

**Definition 2.3.1** A real-valued stochastic process $W(\cdot)$ is called Brownian motion or Wiener process if

1. $W(0) = 0$ a.s.;
2. $W(t) - W(s)$ is $\mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$;
3. for all times $0 < t_1 < t_2 < \ldots < t_n$ the random variables $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent ("independent increments").

Notice in particular that

$$\mathbb{E}[W(t)] = 0, \quad \mathbb{E}[W^2(t)] = t \text{ for each time } t \geq 0.$$

Simulation of $W_t$ is an easy step because we have that $\mathcal{L}(W_t) = \mathcal{N}(0, t)$. Let $\Delta t > 0$ be a constant time increment. For the discrete instances $t_j = j\Delta t$, the value $W_t$ can be written as a sum of increments $\Delta W_k$,

$$W_{j\Delta t} = \sum_{k=1}^{j} \left(W_{k\Delta t} - W_{(k-1)\Delta t}\right).$$
Increments $\Delta W$ can then be calculated from standard normally distributed random numbers $g$. The implication

$$g \sim \mathcal{N}(0, 1) \Rightarrow g \cdot \sqrt{\Delta t} \sim \mathcal{N}(0, \Delta t)$$

leads to the discrete model of a Wiener process

$$\Delta W_k = g \sqrt{\Delta t} \text{ for } g \sim \mathcal{N}(0, 1) \text{ for each } k.$$ 

Therefore, to simulate the value $W_t$ we first generate a Gaussian standard variable $g$ and then we compute $W_t$ as $\sqrt{t}g$.

We now detailed the forward approach to simulate a Brownian path. Typically, we have to simulate $W$ over $T = \{t_k; k = 0, \ldots, M, t_0 = 0, t_M = T\}$.

The Forward Simulation of $W_t$ over $T$ is given by:

$$W(0) = 0,$$
$$W(t_{k+1}) = W(t_k) + \sqrt{t_{k+1} - t_k}g_k,$$

where $(g_1, \ldots, g_M)$ are independent Gaussian standard variables. If we use a discretization with evenly spaced intervals of size $h = \frac{T}{M}$, we have:

$$W(0) = 0,$$
$$W(t_{k+1}) = W(t_k) + \sqrt{h}g_k,$$
2.3.2 Simulations of diffusion in Black and Scholes model

In the Black and Scholes model, underlying asset price $S_t$ follows the diffusion:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2.3.1}$$

and then the price is a geometric Brownian process:

$$S_t = S_0 \exp \left( (r - \frac{\sigma^2}{2}) t + \sigma W_t \right)$$

In this particular case for which we have an explicit solution of the diffusion process, simulation of price paths is based on simulation of Brownian motion described in the last section.

For the discretization $T$, we have:

$$S_{t_{k+1}} = S_{t_k} \exp \left( (r - \frac{\sigma^2}{2})(t_{k+1} - t_k) + \sigma W_{t_{k+1} - t_k} \right)$$

and for a discretization with evenly spaced intervals of size $h$, we simply have:

$$S_{t_{k+1}} = S_{t_k} \exp \left( (r - \frac{\sigma^2}{2})h + \sigma \sqrt{h} g_k \right) \tag{2.3.2}$$
Monte Carlo simulations

Let us now briefly discuss the more general case

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t. \] (2.3.3)

If we don’t have any explicit solution for \( X_t \) (like for Black and Scholes model), we have to use approximation schemes with a discretization of the process. The most known and basic scheme is the Euler scheme. It takes into account a discretization \( T \) of length \( h \); the diffusion (2.3.3) is expressed as:

\[ X_{t_{k+1}} = X_{t_k} + b(X_{t_k})h + \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}), \]

and simulation is obtained with a forward algorithm by:

\[ X_{t_{k+1}} = X_{t_k} + b(X_{t_k})h + \sigma(X_{t_k})\sqrt{hg_k} \]

for \( k = 0, \ldots, M - 1. \)

2.3.3 Simulation of jump-diffusion in Merton model

We first give an idea to obtain a jump-diffusion model for the asset price \( S \) starting from the Black and Scholes dynamic.

Let the price \( S_t \) jumps at random times \( \tau_1, \ldots, \tau_n, \ldots \) and suppose that the relative change in its value at a jump time is given by \( U_1, \ldots, U_n, \ldots \) respectively. We may then assume that, between two jump times, the price \( S_t \) follows the Black and Scholes model; that \( \tau_n \) are the time jumps of a Poisson process \( N_t \) with intensity \( \lambda_t \) and that \( U_n \) is a sequence of i.i.d. random variables that take values in \((-1, +\infty)\). Then, on the intervals \( [\tau_n, \tau_{n+1}) \)

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

while at time \( t = \tau_n \) the jump is given by

\[ \Delta S_n = S_{\tau_n} - S_{\tau_n^-} = U_n S_{\tau_n^-}, \]

so that

\[ S_{\tau_n} = (1 + U_n)S_{\tau_n^-}, \] (2.3.4)

which the assumption \( U_n > -1 \) leads always to positive values of the prices.

We have, for \( t \in [0, \tau_1) \):

\[ S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}, \]

then for \( t \rightarrow \tau_1^- \)

\[ S_{\tau_1^-} = S_0 e^{(\mu - \sigma^2/2)\tau_1 + \sigma W_{\tau_1}}, \]

and

\[ S_{\tau_1} = S_0 (1 + U_1) e^{(\mu - \sigma^2/2)\tau_1 + \sigma W_{\tau_1}}. \]
Figure 2.3: 6 paths of the underlying asset $S_t$ in the jump-diffusion model (2.3.5), for $\lambda = 1$, $\sigma = 0.15$, $r = 0.5$, $U_n = \exp(\gamma + \delta g_n)$, $g_n \sim \mathcal{N}(0,1)$, $\gamma = 0$, $\delta = 0.4$ and $\Delta t = 10^{-2}$.

Moreover, for $t \in [\tau_1, \tau_2]$,

$$S_t = S_{\tau_1} e^{(\mu - \sigma^2/2)(t-\tau_1) + \sigma(W_t-W_{\tau_1})} = S_{\tau_1} (1 + U_n) e^{(\mu - \sigma^2/2)(t-\tau_1) + \sigma(W_t-W_{\tau_1})} = S_0 (1 + U_n) e^{(\mu - \sigma^2/2)t + \sigma W_t}.$$ 

Furthermore, by a recursive argument

$$S_t = S_0 \left( \prod_{n=1}^{N_t} (1 + U_n) \right) e^{(\mu - \sigma^2/2)t + \sigma W_t}. \quad (2.3.5)$$

By the generalized Ito formula (see for instance [58]), the process $S_t$ in (2.3.5) is seen to be a solution of the following stochastic differential equation

$$dS_t = S_t \left( \mu dt + \sigma dW_t + U_t dN_t \right). \quad (2.3.6)$$

This equation corresponds to (2.3.1) with the addition of a jump term and it is a particular case of the general jump-diffusion model.

To simulate this process, we notice that

$$S_{n\Delta t} = S_0 \times \left( S_{\Delta t}/S_0 \right) \times \left( S_{2\Delta t}/S_{\Delta t} \right) \times ... \times \left( S_{n\Delta t}/S_{(n-1)\Delta t} \right). \quad (2.3.7)$$
If we denote $Y_k = \left( \frac{S_k \Delta t}{S_{k-1} \Delta t} \right)$, it is easy to prove that the $(Y_k)_{k \geq 1}$ is a sequence of i.i.d random variables. Then, from (2.3.7), to simulate $S$ at time $n \Delta t$ it is sufficient to simulate $Y_1 = S_{\Delta t}/S_0$. Therefore, we have the following procedure:

- We simulate a Gaussian variable $g \sim \mathcal{N}(0,1)$;
- We simulate a Poisson random variable $N$ with parameter $\lambda \Delta t$;
- If $N=k$, we simulate $k$ independent random variables with law $\mu(dx)$: $U_1, \ldots, U_k$.

### 2.3.4 An application to the European option pricing

In this subsection we deal with pricing call and put European options driven in the jump-diffusion (2.3.6). The underlying asset price is then given by the formula (2.3.5), and the discounted price $(e^{-rt}S_t)_{0 \leq t \leq T}$ is a martingale under the assumption $\mu = r - \lambda \mathbb{E}[U_1]$ (see Proposition 1.2.3).

The price of an European option which guarantees the payoff $\phi(S_T)$ at time $T$ is given by

$$V_0 = \mathbb{E} \left[ e^{-rT} \phi(S_T) \right].$$

By the Theorem 2.1.2, the price estimator is:

$$V_0 \approx \frac{e^{-rT}}{n} \sum_{i=1}^{n} \phi(S_T^{(i)}).$$

To simulate the $n$ values $S_T^{(i)}$ we use the procedure described in the previous sections. If we suppose the jump amplitude $U$ to be log-normal distributed with constant mean $\gamma$ and variance $\delta$, then the $i^{th}$ iteration consists on:

- simulate a Gaussian variable $g_i \sim \mathcal{N}(0,1)$, to compute the value $W_T^{(i)} = g_i \sqrt{T}$ of the Brownian component;
- simulate a Poisson random variable $N_T^{(i)}$ with parameter $\lambda T$;
- For $j = 1, \ldots, N_T^{(i)}$, simulate a Gaussian variable $g_j \sim \mathcal{N}(0,1)$, to compute the value $U_j = \exp (\gamma + \delta g_j)$.

Then,

$$S_T^{(i)} = S_0 e^{(r-\lambda \mathbb{E}[U_1]-\frac{\sigma^2}{2})T} e^{\sigma W_T^{(i)}} \left( \prod_{j=1}^{N_T} e^{\gamma + \delta g_j} \right),$$

where $\mathbb{E}[U_1] = \exp (\gamma + \delta^2/2)$. 

Remark 2.3.2 In this particular case an analytical formula as an infinite sum, for the price of European call and put options may be given [50, 64]. Let define \( \tau = T - t \) and \( \bar{k} = \mathbb{E}[U_1] \),

\[
\begin{align*}
  d_1 &= \frac{\ln \frac{S}{K} + (r_n + \sigma_n \tau)}{\sigma_n \sqrt{\tau}}, \\
  \sigma_n^2 &= \sigma^2 + \frac{\sigma^2_n}{\tau}, \\
  d_2 &= d_1 - \sigma_n \sqrt{\tau}, \\
  r_n &= r - \lambda(\bar{k} + 1) + \frac{n}{\tau} \ln(1 + (\bar{k} + 1)).
\end{align*}
\]

Then, we have

\[
V_t(x) = \sum_{n=0}^{\infty} e^{-\lambda(1+(\bar{k}+1))\tau} \frac{\lambda^n(1 + (\bar{k} + 1))^{n\tau}}{n!} K_n(t, x), \tag{2.3.8}
\]

where

\[
K_n^{\text{call}}(t, x) = xN(d_1) - Ke^{-r_n \tau}N(d_2),
\]

and

\[
K_n^{\text{put}}(t, x) = Ke^{-r_n \tau}N(-d_2) - xN(-d_1),
\]

with

\[
N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx.
\]

To end this section, in Figure 2.4, we present the value of the call option given respectively by the jump-diffusion model (“C-Jump”), and the pure diffusion model (“C-Diffusion”). We fix the maturity \( T = 10 \) and we compute the price \( V_t(S) \) at the exercise price \( S = K \) for \( t \in [0, T] \). The differences between the two models is clear: the price in the jump diffusion model is larger than the one in the pure diffusion setting, according to the theoretical results in [50]. Similar results are observed in Figure 2.5, for the put option prices.
Figure 2.4: Call option prices $V_t(S)$ at the exercise price $S = K = 100$ varying the time $t$ from today to maturity $T = 10$, driven respectively in the jump-diffusion market (“C-Jump”) and in the diffusion market (“C-Diffusion”).
Figure 2.5: Put option prices $V_t(S)$ at the exercise price $S = K = 100$ varying the time $t$ from today to maturity $T = 10$, driven respectively in the jump-diffusion market ("P-Jump") and in the diffusion market ("P-Diffusion").
Chapter 3

Viscosity solutions for second order partial differential equations

In this chapter we shall present some backgrounds on the general theory of viscosity solutions for fully nonlinear possibly degenerate differential parabolic equations.

First, we deal with Cauchy problems of the form:

\[ \partial_t u + F(x, t, u, Du, D^2u) = 0, \quad (x, t) \in \mathbb{R}^n \times (0, T) \]  

\[ u(x, 0) = u_0(x) \quad u_0 \in C(\mathbb{R}^n). \]

The first definition of viscosity solutions has been given by Crandall and Lions [25] for first order Hamilton–Jacobi equations; starting from that paper, several other results for more general equations have been obtained. The viscosity solutions theory comes out to be a successful tool as it allows merely continuous function to be solution of fully nonlinear possibly degenerate second order differential problems. This theory applies to partial differential equations (3.0.1) where \( F \in C(\mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times S_n; \mathbb{R}) \) satisfies the following properties:

**F.1** \( F \) is degenerate elliptic, namely

\[ F(x, t, u, \mathcal{X}) \geq F(x, t, u, \mathcal{Y}) \text{ if } \mathcal{X} \leq \mathcal{Y}, \]

**F.2** \( F \) is quasi-monotone as a function of \( u \), uniformly with respect to the other variables, namely

\[ F(x, t, u, p, \mathcal{X}) \geq F(x, t, v, p, \mathcal{X}) - \gamma(u - v) \text{ whenever } u \geq v. \]

Here \( \gamma \in C([0, \infty)) \), \( \gamma(0) = 0 \), \( \gamma(u) > 0 \) as \( u > 0 \) and

\[ \int_0^\varepsilon \frac{du}{\gamma(u)} = \infty \quad \text{for all } \varepsilon > 0. \]
An operator $F$ of such kind satisfying $F.1$ and $F.2$ is referred to as a Hamilton–Jacobi operator. Assumption $F.2$, is also known as Osgood’s condition and it is the parabolic version of the properness condition in the elliptic case, that is

$$F(x, u, p, \mathcal{X}) \geq F(x, v, p, \mathcal{X})$$

whenever $u \geq v$.

An overview of viscosity solutions results for second order partial differential equation can be found in the User’s guide [23].

The discussion is then extended to the case of integro–differential equations,

$$\partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2u) = 0.$$  \hfill (3.0.3)

The term $\mathcal{I}u$ is an integral operator of the form

$$\mathcal{I}u(x, t) = \int_{\mathbb{R}^N} M(u(x + z, t), u(x, t))\mu_{x,t}(dz),$$

with $M \in C(\mathbb{R}^2; \mathbb{R})$ nondecreasing in its first argument, $\mu_{x,t}(dz)$ a positive Radon measure on $\mathbb{R}^N$ for all $(x, t)$.

Beside the general assumptions $F.1$ and $F.2$, to get similar results of the pure differential problem, it has to be assumed on $F$ the following additional condition:

$\text{F.3}$ $F$ is non-increasing with respect to $\mathcal{I}u$, namely

$$F(x, t, u, I, p, \mathcal{X}) \geq F(x, t, u, J, p, \mathcal{X})$$

if $I \leq J$.

For an overview and for more precise statements and proofs of viscosity solutions results for second order partial integro-differential equations, we refer to [5].

Finally, notice that the solution of problems (3.0.1) and (3.0.3) is related to the price of an European derivative with payoff $\phi(X_T) = u_0(X_T)$ at maturity $T$, written on an incomplete market driven by a diffusion process and a jump-diffusion process respectively (see Section 1.2.3 and reference therein).

### 3.1 Purely differential problems

In this section we shall give an overview of viscosity solution theory for the Cauchy problem (3.0.1)–(3.0.2), which is proven to be well posed by means of the Perron’s method and the comparison principle. Detailed proofs can be found in [23].

The viscosity solution theory starts from the definition of the so called parabolic semijets which allows a definition of $\partial_t u$, $\mathcal{D}u$ and $\mathcal{D}^2u$ for merely continuous function. Two equivalent definitions can be given, one local and one global.
Definition 3.1.1 (parabolic semijet: local definition) We say that $(\tau, p, \mathcal{X}) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n$ belongs to $\mathcal{P}^\pm u(x, t)$, the parabolic superjet of $u$ at the point $(x, t)$, if
\[
u(y, s) \leq u(x, t) + \tau(t - s) + p \cdot (x - y) + \frac{1}{2} \mathcal{X}(x - y) \cdot (x - y)
+ o(|t - s| + \|x - y\|^2),
\]
as $\mathbb{R}^n \times [0, T) \ni (y, s) \to (x, t)$. Here $o$ stands for a modulus of continuity.

We say that $(\tau, p, \mathcal{X}) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n$ belongs to $\mathcal{P}^\pm u(x, t)$, the parabolic subjet of $u$ at the point $(x, t)$ if $-(\tau, p, \mathcal{X}) \in \mathcal{P}^\pm (-u)(x, t)$.

We define the closed parabolic semijets the set $\overline{\mathcal{P}^\pm}$ and $\overline{\mathcal{P}^-}$ given as
\[
\overline{\mathcal{P}^\pm} u(x, t) = \left\{ (\tau, p, \mathcal{X}) : \text{there are } (x_n, t_n) \to (x, t) \text{ and } (\tau_n, p_n, \mathcal{X}_n) \in \mathcal{P}^\pm u(x_n, t_n), (\tau_n, p_n, \mathcal{X}_n) \to (\tau, p, \mathcal{X}) \right\}.
\]

Remark 3.1.2 This definition is equivalent to requiring that there exists a function $\phi \in C^{2,1}(\mathbb{R}^n \times [0, T))$ such that $u - \phi$ has a local maximum (resp. minimum) at $(x, t)$ and $(\tau, p, \mathcal{X}) = (\partial_t \phi, \mathcal{D} \phi, \mathcal{D}^2 \phi)(x, t)$.

It can be noticed that replacing $\phi(y, s)$ by
\[
\phi(y, s) + (t - s)^2 + \|x - y\|^4 + [u(x, t) - \phi(x, t)],
\]
\[
\phi(y, s) - (t - s)^2 - \|x - y\|^4 - [u(x, t) - \phi(x, t)],
\]
the local maximum (resp. minimum) is a strict maximum (resp. minimum). □

We now state the global definition of the semijets.

Definition 3.1.3 (parabolic semijet: global definition) We call a good test function for the superjet (resp. subjet) of $u$ at $(x, t)$ any function $\phi \in C^{1,2}(\mathbb{R}^n \times [0, T))$ such that $u - \phi$ has a global strict maximum (resp. minimum) at $(x, t)$ with $u(x, t) = \phi(x, t)$.

Afterwards, we define $(\tau, p, \mathcal{X}) \in \mathcal{P}^\pm u(x, t)$ if there exist a good test function $\phi$ for the superjet (resp. subjet) of $u$ at $(x, t)$ such that
\[(\tau, p, \mathcal{X}) = (\partial_t \phi, \mathcal{D} \phi, \mathcal{D}^2 \phi)(x, t).
\]

An important continuity property can be proved for the semijets.

Lemma 3.1.4 Let $u, u_n \in \mathcal{USC}$ (resp. $\mathcal{LSC}$) and $(x, t) \in \mathbb{R}^n \times (0, T)$ such that $u_n \leq u$ (resp. $u_n \geq u$) pointwise near $(x, t)$ and there exists a sequence $(y_n, s_n)$ converging to $(x, t)$ such that
\[
\lim_{n \to \infty} u_n(y_n, s_n) = u(x, t).
\]
Then for all \((\tau, p, \mathcal{X}) \in \mathcal{P}^+ u(x, t)\) there exists a sequence \((x_n, t_n)\) converging to \((x, t)\) and \((\tau_n, p_n, \mathcal{X}_n) \in \mathcal{P}^+ u(x_n, t_n)\) (resp. \(\mathcal{P}^-\)) such that

\[
\lim_{n \to \infty} u(x_n, t_n) = u(x, t), \quad (3.1.1)
\]

\[
\lim_{n \to \infty} (\tau_n, p_n, \mathcal{X}_n) = (\tau, p, \mathcal{X}). \quad (3.1.2)
\]

**Definition 3.1.5** Given a function \(u\) and a point \((x, t) \in \mathbb{R}^n \times [0, T)\), we say that

\[
\partial_t u(x, t) + F(x, t, u, D_u, D^2 u) \leq 0, \quad (\text{resp. } \geq 0)
\]

in viscosity sense at \((x, t)\) if one of the following equivalent conditions is met:

(i) for all \((\tau, p, \mathcal{X}) \in \mathcal{P}^+ u(x, t)\) (resp. \(\mathcal{P}^- u(x, t)\))

\[
\tau + F(x, t, u(x, t), p, \mathcal{X}) \leq 0, \quad (\text{resp. } \geq 0)
\]

(ii) for each function \(\phi \in C^{1,2}(\mathbb{R}^n \times [0, T))\) such that \(u - \phi\) has a local maximum (resp. minimum) point at \((x, t)\)

\[
\partial_t \phi(x, t) + F(x, t, u(x, t), D\phi(x, t), D^2\phi(x, t)) \leq 0, \quad (\text{resp. } \geq 0)
\]

(iii) for each good test function \(\phi \in C^{1,2}(\mathbb{R}^n \times [0, T))\) touching from above (resp. below) the graph of \(u\) at \((x, t)\)

\[
\partial_t \phi + F(x, t, \phi, D\phi, D^2\phi) \leq 0, \quad (\text{resp. } \geq 0)
\]

holds in classical sense.

When applied to semicontinuous functions, the notion of differential inequality in viscosity sense has an important semicontinuity property, see [23]. Now, we can introduce the viscosity solutions.

**Definition 3.1.6** A function \(u \in U\mathcal{S}C\) is a viscosity subsolution (resp. \(u \in L\mathcal{S}C\) is a viscosity supersolution) of equation (3.0.1) if

\[
\partial_t u + F(x, t, u, D_u, D^2 u) \leq 0, \quad (\text{resp. } \geq 0)
\]

in viscosity sense for all \((x, t) \in \mathbb{R}^n \times (0, T)\).

If, in addition

\[
u(x, 0) \leq u_0(x), \quad (\text{resp. } \geq)
\]

for all \(x \in \mathbb{R}^n\), then \(u\) is a viscosity subsolution (resp. supersolution) of the Cauchy problem (3.0.1)–(3.0.2).

An arbitrary function \(u : \mathbb{R}^n \times [0, T) \to \mathbb{R}\) is a weak viscosity solution for the Cauchy problem (3.0.1)–(3.0.2) if its upper/lower semicontinuous envelopes are sub/supersolutions, respectively.

It is a strong viscosity solution for the Cauchy problem (3.0.1)–(3.0.2) if it is both a sub/supersolution.
By this definition it easily follows that a viscosity solution of the Cauchy problem (3.0.1)–(3.0.2) is continuous on $\mathbb{R}^n \times [0, T)$ and $u(x, 0) = u_0(x)$ for all $x \in \mathbb{R}^n$.

**Remark 3.1.7** If $F$ is continuous in all its arguments, as we shall always suppose in the following, the viscosity inequality in the definition can be replaced by

$$\tau + F(x, t, u, p, X) \leq 0, \text{ (resp.} \geq 0)$$

for all $(\tau, p, X) \in \overline{P}^+ u(x, t) (\text{resp.} \overline{P}^-)$,

at all $(x, t) \in \mathbb{R}^n \times (0, T)$.

With these instruments it is possible to show the well posedness of the Cauchy problem (3.0.1)–(3.0.2). Classical theory of viscosity solution proved existence and uniqueness of viscosity solution for this problem in the class of continuous function with linear growth at infinity. Unfortunately this class does not reflects the needs of financial problems, where typically functions present exponential growth at infinity. This extension was proved by Amadori [5, 2, 4]. Here we present these results for the Cauchy problem using the Perron’s method.

### 3.1.1 Well posedness for the Cauchy Problem

In this subsection we recall the topic of well posedness of the Cauchy problem, starting from the results by Barles and Perthame [10], where a relaxed notion of the initial condition is introduced. Particular attention is given to the extension to exponential growing solution.

**Definition 3.1.8** A function $u \in \mathcal{USC}$ is a subsolution of the problem (3.0.1)–(3.0.2) with [generalized initial condition](#) (GIC) if the initial condition $u(\cdot, 0) \leq u_0(\cdot)$ is replaced by

$$\min\{\partial_t + F(x, 0, u, Du, D^2 u), u - u^0\} \leq 0,$$

in viscosity sense at $t = 0$.

A function $u \in \mathcal{LSC}$ is a supersolution of the Cauchy problem (3.0.1)–(3.0.2) with [generalized initial condition](#) (GIC) if the initial condition $u(\cdot, 0) \geq u_0(\cdot)$ is replaced by

$$\max\{\partial_t + F(x, 0, u, Du, D^2 u), u - u^0\} \geq 0,$$

in viscosity sense at $t = 0$.

This definition can be interpreted as follows: the possibility that the condition $u(x, 0) > u^0(x)$ holds at some $x$ is allowed, but in that case $u$ must be a subsolution till the boundary, that means that

$$\partial_t u + F(x, 0, u, Du, D^2 u) \leq 0, \text{ in viscosity sense at } (x, 0).$$
The same holds for the supersolution case. A function $u$ that is a sub/supersolution with GIC is a sub/supersolution in the sense of Definition 3.1.6, and it has been proved in [10].

We introduce now the Perron’s method to achieve the existence of viscosity solution: it is based on the construction of a monotone approximation scheme and it allows to gain existence of weak viscosity solution once continuity of sub/supersolution is proved. This will result a central point in the proof of the numerical result in Chapter 5.

**Proposition 3.1.9** Let $\{u_n\}_{n \in \mathbb{N}} \in \mathcal{USC}$ (resp. $\mathcal{LSC}$) be an increasing (resp. decreasing) sequence of subsolution (resp. supersolution) of equation (3.0.1) such that $u_n$ converges pointwise to $u$. If the limit function $u$ is locally bounded, then it is a subsolution (resp. supersolution) of equation (3.0.1).

This result follows easily from the continuity property of the semijets.

We can present now the main result for the existence of solutions, see [35] for the proof.

**Theorem 3.1.10 [Perron’s Method]** Let $h, k \in C(\mathbb{R}^n \times [0, T))$ be a viscosity subsolution and supersolution respectively of (3.0.1)–(3.0.2) such that $h \leq k$ pointwise on $\mathbb{R}^n \times [0, T)$. Let now define the set

$$S = \{v : v \text{ is a subsolution of } (3.0.1)–(3.0.2), h \leq v \leq k \text{ pointwise on } \mathbb{R}^n \times [0, T)\}$$

and the function

$$u(x, t) = \sup \{v(x, t) : v \in S\}. \quad (3.1.3)$$

Then $u$ is a viscosity solution for (3.0.1)–(3.0.2).

From Theorem 3.1.10 no continuity property can be deduced for the solution $u$ given by (3.1.3), but once the continuity is proved, $u$ is a strong viscosity solution of the Cauchy problem.

We can now state the well posedness of the Cauchy problem in terms of the comparison principle, see [23] for a proof based on upper and lower envelopes.

**Proposition 3.1.11** Let $g \in C[0, \infty))$ be a rate of growth and assume that

1. for all initial data $u_0 \in C(\mathbb{R}^n) \cap L^\infty_g(\mathbb{R}^n)$, the Cauchy problem (3.0.1)–(3.0.2) admits a subsolution $h$ and a supersolution $k$ in $C(\mathbb{R}^n \times [0, T)) \cap L^\infty(0, T; L^\infty_g(\mathbb{R}^n))$ such that $h \leq k$ pointwise;

2. for any $u \in \mathcal{USC} \cap L^\infty(0, T; L^\infty_g(\mathbb{R}^n))$ subsolution, and for any $v \in \mathcal{LSC} \cap L^\infty(0, T; L^\infty_g(\mathbb{R}^n))$ of (3.0.1)–(3.0.2) there holds

$$u \leq v, \text{ pointwise,}$$

that is a comparison principle among sub/supersolutions holds.
Then the Cauchy problem (3.0.1)–(3.0.2) is well posed in the class $L^\infty_0(\mathbb{R}^n)$: for all initial data $u_0 \in C(\mathbb{R}^n) \cap L^\infty_0(\mathbb{R}^n)$ there exists an unique viscosity solution of the Cauchy problem in $C(\mathbb{R}^n \times [0,T)) \cap L^\infty(0,T; L^\infty_0(\mathbb{R}^n))$, and it is given by (3.1.3).

From this results we can see that choosing a suitable class of growth and using a comparison principle, the well posedness of the Cauchy problem can be obtained quite easily. In the case of Cauchy problems related to the price of an European derivative, the typical rate of growth is exponential, because of the logarithmic change of variable, as it has been explained in Chapter 1. Moreover the presence of nonlinearities due to the presence of large investor in the market requires the class of growth to be enlarged, under proper assumption on $F$.

The standard comparison principle reads, see [23], Theorem 3.1.12 Let us suppose that $u \in USC \cap L^\infty(0,T; L^\infty_{lin,+}(\mathbb{R}^n))$ and $\overline{u} \in LSC \cap L^\infty(0,T; L^\infty_{lin,-}(\mathbb{R}^n))$ are respectively a subsolution and supersolution of the Cauchy problem (3.0.1)–(3.0.2). Then $u \leq \overline{u}$ pointwise in $\mathbb{R}^n \times [0,T)$.

To extend this result to the class of growth that is of interest for financial applications, we can split the $F$ operator in two parts, one containing all the linear terms, the other one containing the nonlinear first order term:

$$F(x, t, u, Du, D^2u) = \mathcal{L}(x, t, uD, D^2u) + H(x, t, u, Du),$$

where $\mathcal{L}$ is a linear degenerate second order operator

$$\mathcal{L}(x, t, u, D^2u)u = -\frac{1}{2}\text{tr}[aa^T D^2u] + bDu + cu,$$

$a = (a^i_j)$ is a $n \times d$ matrix, with $d \leq n$ and

$$a^i_j, b_i, c \in C(\mathbb{R}^n \times [0,T]) \cap L^\infty(\mathbb{R}^n \times (0,T)),
\mathcal{L}(1)$$

$$a^i_j, b_i \in L^\infty(0,T; W^{1,\infty}_{loc}(\mathbb{R}^n)).$$

$H \in C(\mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ is a nonlinear first order operator such that

$$(H.1) \quad H(\cdot, \cdot, 0, 0) \in L^\infty(0,T; L^\infty_{\text{exp}}(\mathbb{R}^n)),$$

that is that $H(\cdot, \cdot, 0, 0)$ plays the role of a source term and has the same rate of growth of the initial data.

Let us suppose that for each $R > 0$ there exists a modulus of continuity $\omega_R$ and a constant $l_R > 0$ such that

$$(H.2.i) \quad |H(x, t, u, p) - H(y, t, u, p)|$$

$$\leq (1 + |u|)\omega_R(\|x - y\|) + l_R(1 + \|p\|)\|x - y\|,$$
for all $x, y \in B(0, R)$. Moreover there exists a constant $l' \geq 0$ such that
\[(H.2.ii) \quad H(x, t, u, p) - H(x, t, v, q) \leq l' [ |u - v| + \|p - q\|].\]
it can be proved that assumption $(L.1)$ and $(H.2.ii)$ imply that assumption $F.2$ is verified with
\[\gamma(u) = \max\{L' - \inf c, 0\} u\]
If these assumptions hold, then a comparison result for exponential growing solution can be derived.

**Theorem 3.1.13** [5, Theorem 3.20, page 87] Let us assume that $u_0 \in C(\mathbb{R}^n) \cap L^\infty_{\exp}(\mathbb{R}^n)$; let $\underline{u} \in \text{USC} \cap L^\infty(0, T; L^\infty_{\exp}(\mathbb{R}^n))$ and $\bar{u} \in \text{LSC} \cap L^\infty(0, T; L^\infty_{\exp}(\mathbb{R}^n))$ be respectively subsolution and supersolution of the Cauchy problem (3.0.1)–(3.0.2). Then
\[\underline{u} \leq \bar{u}, \text{ pointwise on } \mathbb{R}^n \times [0, T).\]
A detailed proof can be found in [5]: it is based on the choice of a good weight function by which multiply $u$ and $\bar{u}$. A particular care is needed to deal with nonlinear terms.

**Financial applications.** In the previous section we have seen that the class of existence of viscosity solutions for the Cauchy problem depends on the class of growth of the initial datum. For what concerns the Cauchy problem related to the pricing of an European derivative, the initial datum belongs to $C(\mathbb{R}^n) \cap L^\infty_{\exp}(\mathbb{R}^n)$. It holds the following:

**Corollary 3.1.14** Under the assumptions $(L.1)$–$(H.1)$ the Cauchy problem (3.0.1)–(3.0.2) is well posed in the class $C(\mathbb{R}^n) \cap L^\infty_{\exp}(\mathbb{R}^n)$ in the framework of viscosity solutions, that is for all $u_0 \in C(\mathbb{R}^n) \cap L^\infty_{\exp}(\mathbb{R}^n)$ there exists a viscosity solution $u$ such that
\[|u(x, t)| \leq ce^{n|x|},\]
with
\[n = \min\{m \in \mathbb{N} : e^{-m||x||}u_0(x), e^{-m||x||}H(x, t, 0, 0) \text{ are bounded}\}.\]
The obtained solution is unique in the class $L^\infty(0, T; L^\infty_{\exp}(\mathbb{R}^n))$.

### 3.2 The Integro-Differential case

As we stated in the introduction to this chapter and in Section 1.2.3, the price of an European derivative with payoff $\phi(X_T) = u_0(X_T)$ at maturity $T$, written on an incomplete market driven by a jump-diffusion process, may be characterized by the solution of a Cauchy integro-differential problem of the following form:
\[\partial_t u + F(x, t, u, Lu, Du, D^2u) = 0 \quad (x, t) \in \mathbb{R}^n \times (0, T), \quad (3.2.1)\]
where $F \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \times S_N, \mathbb{R})$, $S_N$ is the set of the symmetric $D \times D$ matrices and

$$
Iu(x,t) = \int_{\mathbb{R}^N} M(u(x+z,t),u(x,t))\mu_{x,t}(dz),
$$

In Section 1.2.2, the linear version of these problems has been analyzed, and a result of existence and uniqueness of the solution in the classical sense has been given. Actually, due to the possible degeneracy of the parabolic operator, which is typical of incomplete markets, and the presence of possible nonlinear terms, as in a large investor economy, we need to study weaker solutions. Here, we extend the viscosity solutions approach given in the previous sections for the purely differential case, to (3.2.1)-(3.2.2). For more precise statements and proofs of viscosity solutions results for second order partial integro-differential equations, we refer to [5].

For completeness of the exposition, we recall here the assumptions on $F$ given in the introduction of this chapter.

**F1** $F$ is degenerate elliptic:

$$
F(x,t,u,\mathcal{I},p,\mathcal{X}) \geq F(x,t,u,\mathcal{I},p,\mathcal{Y}), \text{ for } \mathcal{X} \leq \mathcal{Y};
$$

**F2** $F$ is quasi-monotone with respect to $u$, uniformly with respect to the other variables:

$$
F(x,t,u,\mathcal{I},p,\mathcal{X}) \geq F(x,t,v,\mathcal{I},p,\mathcal{X}) - \gamma(u - v), \text{ for } u \geq v,
$$

where $\gamma \in C([0,\infty))$, $\gamma(0) = 0$, $\gamma(u) > 0$ as $u > 0$ and:

$$
\int_0^\varepsilon \frac{du}{\gamma(u)} = \infty \text{ for all } \varepsilon > 0.
$$

**F3** $F$ is non-increasing with respect to $\mathcal{I}$:

$$
F(x,t,u,\mathcal{I},p,\mathcal{X}) \geq F(x,t,u,\mathcal{J},p,\mathcal{X}), \text{ for } \mathcal{I} \leq \mathcal{J}.
$$

Following [1, 5, 3], we can now give a modified notion of viscosity solutions, which makes use of the notion of upper and lower semijets $\mathcal{P}^\pm$ (see Definition 3.1.1).

**Definition 3.2.1** Given a function $u$ and a point $(x,t) \in \mathbb{R}^N \times [0,T)$, we say that:

$$
\partial_t u + F(x,t,u,\mathcal{I}u,\nabla u,\nabla^2 u) \leq 0 \text{ (resp. } \geq 0)
$$

in viscosity sense at $(x,t)$ if one of the following equivalent condition holds:
1. For all \((\tau, p, X) \in \mathcal{P}^+(u(x, t)) \) (respectively \(\mathcal{P}^-(u(x, t))\)):
\[
\tau + F(x, t, u(x, t), \mathcal{I}u(x, t), p, X) \leq 0 \text{ (resp. } \geq 0) ;
\]

2. For each function \(\phi \in C^{2,1}(\mathbb{R}^N \times [0, T])\) such that \(u - \phi\) has a local maximum at \((x, t)\) (respectively, a minimum):
\[
\partial_t \phi(x, t) + F(x, t, u(x, t), \mathcal{I}\phi(x, t), \mathcal{D}\phi(x, t), \mathcal{D}^2\phi(x, t)) \leq 0 \text{ (resp. } \geq 0) ;
\]

3. For all test function \(\phi \in C^{2,1}(\mathbb{R}^N \times [0, T])\) such that \(u - \phi\) has a global strict maximum at \((x, t)\) (respectively, a minimum) and \(\phi(x, t) = u(x, t)\):
\[
\partial_t \phi + F(x, t, \phi, \mathcal{I}\phi, \mathcal{D}\phi, \mathcal{D}^2\phi) \leq 0 \text{ (resp. } \geq 0) \]
holds in classical sense.

Let us notice that, the important difference between the purely differential case, treated in the previous sections, and the integro-differential one is that the local continuity property of the semijets of semicontinuous functions Lemma 3.1.4, does not imply the semicontinuity of the equation, because the new nonlocal term \(\mathcal{I}u\) does not preserve semicontinuity in general. To overcome this difficulty, it is necessary to define a new class of admissible functions.

**Definition 3.2.2** A function \(f(y, s; z)\) has an **upper** (resp. **lower**) \(\mu\)-bound at \((x, t)\) if there exist a neighborhood \(V_{x,t}\) of \((x, t)\) and a function \(\Phi \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N; \mu_{x,t})\) such that:

- \(\int \Phi(z) \mu_{y,s}(dz) \to \int \Phi(z) \mu_{x,t}(dz) \) if \((y, s) \to (x, t)\);

- \(f(y, s; z) \leq \Phi(z) \) (resp. \( \geq \)) \(\mu_{y,s} - \) a.e. \(z\), for all \((y, s) \in V_{x,t}\).

**Remark 3.2.3** Let \(f\) be a locally bounded function on \(\mathbb{R}^N \times [0, T) \times \mathbb{R}^N\) which has an upper (resp., lower) \(\mu\)-bound at \((x, t)\), then (see [1]):

\[
\limsup_{(y, s) \to (x, t)} \int_{\mathbb{R}^N} f(y, s; z) \mu_{y,s}(dz) \leq \int_{\mathbb{R}^N} \limsup_{(y, s) \to (x, t)} f(y, s; z) \mu_{x,t}(dz)
\]

(respectively, \(\liminf\))

\[
\left( \liminf_{(y, s) \to (x, t)} \int_{\mathbb{R}^N} f(y, s; z) \mu_{y,s}(dz) \geq \int_{\mathbb{R}^N} \liminf_{(y, s) \to (x, t)} f(y, s; z) \mu_{x,t}(dz) \right).
\]

In this way we can introduce some new classes of admissible functions.

**USC** \(^{T}\) is the set of upper semicontinuous, locally bounded functions on \(\mathbb{R}^N \times [0, T)\) such that \(M(u(x + z, t), u(x, t))\) has an upper \(\mu\)-bound at any \((x, t)\);
\( \mathcal{LSC}^I \) is the set of lower semicontinuous, locally bounded functions on \( \mathbb{R}^N \times [0, T) \) such that \( M(u(x+z,t),u(x,t)) \) has a lower \( \mu \)-bound at any \((x,t)\);

\( \mathcal{C}^I = \mathcal{USC}^I \cap \mathcal{LSC}^I \).

It can be observed that if the integral operator \( \mathcal{I} \) has some more regularities, the admissible classes described before coincide with the classes of exponential growth at infinity, that are the classes of growth required in the framework of pure diffusion models; for details we refer to [5, 3].

**Lemma 3.2.4** Let \( \mathcal{I} \) be an integral operator:

\[
\mathcal{I} u(x,t) = \int_{\mathbb{R}^N} M(u(x+z,t),u(x,t)) \mu_{(x,t)}(dz)
\]

satisfying:

\begin{enumerate}
\item \( F.4 \) \( M \in \mathcal{C}(\mathbb{R}^2) \), \( M(0,0) = 0 \) and there exist \( M \geq 0 \) such that:
\[
M(a,b) - M(c,d) \leq M[(a-c)^+ + |b-d|],
\]

\item \( F.5 \) for all \((x,t)\), \( \mu_{(x,t)} \) is a positive bounded measure on \( \mathbb{R}^N \) and there is a rate of growth \( g \) such that the function:
\[
(x,t) \rightarrow \int_{\mathbb{R}^N} [1 + g(||z||)] \mu_{(x,t)}(dz)
\]

is continuous on \( \mathbb{R}^N \times [0, T) \).
\end{enumerate}

Then the following inclusion holds:

\[
\mathcal{USC} \cap L^\infty(0,T;L^\infty(\mathbb{R}^N)) \subseteq \mathcal{USC}^I \quad \mathcal{LSC} \cap L^\infty(0,T;L^\infty(\mathbb{R}^N)) \subseteq \mathcal{LSC}^I
\]

Now, with these results, we can define viscosity sub/super solutions for integro-differential equations:

**Definition 3.2.5** \( u \in \mathcal{USC}^I \) (\( u \in \mathcal{LSC}^I \)) is a **viscosity subsolution** (resp. **viscosity supersolution**) of the equation (3.2.1) if:

\[
\partial_t u + F(x,t,u,Iu,Du,D^2u) \leq 0 \quad \text{(resp.} \geq)\]

holds in viscosity sense for all \((x,t)\in\mathbb{R}^N \times (0,T)\).

If in addition:

\[
u(x,0) \leq u_0(x) \quad \text{(resp.} \geq)\]

for all \( x \in \mathbb{R}^N \), then \( u \) is a viscosity subsolution (resp. viscosity supersolution) of the integro-differential Cauchy problem (3.2.1)-(3.2.2). An arbitrary function \( u : \mathbb{R}^N \times [0,T) \rightarrow \mathbb{R} \) is a **weak viscosity solution** for the problem (3.2.1)-(3.2.2) if its upper/lower semicontinuous envelopes belong to \( \mathcal{USC}^I/\mathcal{LSC}^I \), respectively, and they are sub/supersolutions. Besides, it is a (strong) **viscosity solution** for the problem (3.2.1)-(3.2.2) if it is both a sub/super solution.

Notice that the viscosity solution of the Cauchy problem is continuous on \( \mathbb{R}^N \times [0,T) \).
3.2.1 Well posedness for the integro-differential Cauchy Problem

In this section we shall recall Perron method and comparison principles to give a result of existence, uniqueness and regularity for the solutions of the integro-differential Cauchy problem (3.2.1)-(3.2.2). The well posedness of this problem shall be given in the class of continuous functions with exponential growth at infinity.

Notice that the notion of initial condition can be relaxed also in this integro-differential setting. All the results of the present subsection are proven in [5].

**Definition 3.2.6** A function \( u \in USC^T \) is a subsolution of the problem (3.2.1)-(3.2.2) with **generalized initial condition (GIC)** if the initial condition \( u(\cdot, 0) \leq u_0(\cdot) \) is replaced by

\[
\min \{ \partial_t + F(x, 0, u, \mathcal{I}u, Du, D^2u), u - u^0 \} \leq 0,
\]

in viscosity sense at \( t = 0 \).

A function \( u \in LSC^T \) is a supersolution of the Cauchy problem (3.2.1)-(3.2.2) with **generalized initial condition (GIC)** if the initial condition \( u(\cdot, 0) \geq u_0(\cdot) \) is replaced by

\[
\max \{ \partial_t + F(x, 0, u, \mathcal{I}u, Du, D^2u), u - u^0 \} \geq 0,
\]

in viscosity sense at \( t = 0 \).

A sub/supersolution with GIC is indeed a sub/supersolution in the sense of Definition 3.2.5.

As done for the pure differential case, we introduce the Perron’s method to achieved the existence of viscosity solutions.

**Proposition 3.2.7** Let \( \{ u_n \}_{n \in \mathbb{N}} \in USC^T \) (resp. \( LSC^T \)) be an increasing (resp. decreasing) sequence of subsolution (resp. supersolution) of equation (3.2.1) and set

\[
u(x, t) = \sup_n u_n(x, t) \quad (respectively, \inf_n).
\]

If \( u \) is locally bounded and it has upper (respectively, lower) \( \mu \)-bounds everywhere, then it is a subsolution (respectively, a supersolution) of (3.2.1).

**Theorem 3.2.8** [Perron’s method] Let \( h \) and \( k \in C(\mathbb{R}^N \times [0, T]) \) be respectively a viscosity sub/supersolution of (3.2.1)-(3.2.2) such that:

(i) \( h \leq k \) pointwise on \( \mathbb{R}^N \times [0, T) \),

(ii) for every \((x, t) \in \mathbb{R}^N \times [0, T)\), there exists a neighborhood \( V \) of \((x, t)\) and a function \( \Phi \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, \mu_x) \) such that

\[
\int \Phi(z) \mu_{ys}(dz) \rightarrow \int \Phi(z) \mu_{xt}(dz)
\]

where \( \mu_{ys}(dz) \) and \( \mu_{xt}(dz) \) are the measures corresponding to \( y \) and \( x \) respectively.
as \((y, s)\) converges to \((x, t)\) and
\[-\Phi(z) \leq M(h(y + z, s), a) \leq M(k(y + z, s), a) \leq \Phi(z)\]
for all \(z \in \mathbb{R}^N\), \((y, s) \in V\) and \(a \in [h(y, s), k(y, s)]\).

We denote by \(S\) the set of subsolutions \(v\) of (3.2.1)-(3.2.2) such that \(h \leq v \leq k\) on \(\mathbb{R}^N \times [0, T]\) and by
\[u(x, t) = \sup\{v(x, t) : v \in S\}.\] (3.2.3)
Then \(u\) is a weak viscosity solution of the integro-differential Cauchy problem (3.2.1)-(3.2.2).

Let us state now the well posedness of the Cauchy problem:

**Proposition 3.2.9** Let \(g \in C([0, +\infty))\) be a rate of growth and assume that

1. for all initial data \(u_0 \in C(\mathbb{R}^N) \cap L^\infty_g(\mathbb{R}^N)\), the integro–differential Cauchy problem (3.2.1)-(3.2.2) admits a subsolution \(h\) and a supersolution \(k\) in \(C(\mathbb{R}^N \times [0, T]) \cap L^\infty(0, T)\) such that items (i) and (ii) of Theorem 3.2.8 hold;
2. for any \(u \in USC^T \cap L^\infty(0, T; L^\infty(\mathbb{R}^N))\) subsolution, and for any \(v \in LSC^T \cap L^\infty(0, T; L^\infty(\mathbb{R}^N))\) of (3.2.1)-(3.2.2) there holds
\[u \leq v, \text{ pointwise,}\]
that is a comparison principle among sub/supersolutions holds.

Then the integro–differential Cauchy problem (3.2.1)-(3.2.2) is well posed in the class \(L^\infty_g(\mathbb{R}^N)\): for all initial data \(u_0 \in C(\mathbb{R}^N) \cap L^\infty_g(\mathbb{R}^N)\) there exists an unique viscosity solution of the integro–differential Cauchy problem in \(C(\mathbb{R}^N \times [0, T]) \cap L^\infty(0, T; L^\infty_g(\mathbb{R}^N))\), and it is given by (3.2.3).

We now state a comparison principle allowing exponential growth at infinity. As it has been done for the pure-differential case, we shall assume that the integro-differential operator \(F\) splits into
\[F(x, t, u, \mathcal{I}u, D\mathcal{D}u, D^2u) = L_I(x, t, u, \mathcal{I}u, D\mathcal{D}u, D^2u) + H_I(x, t, u, \mathcal{I}u, D\mathcal{D}u),\] (3.2.4)
where \(L\) is a linear degenerate-elliptic integro-differential operator,
\[L_I(x, t, u, \mathcal{D}, D^2u)u = -\frac{1}{2}\text{tr}\left[\mathbf{a}a^T D^2u\right] + bD\mathcal{D}u + cu - d\mathcal{I}u,\]
\(a = (a^i_j)\) is a \(N \times d\) matrix, with \(d \leq N\) and
\[a^i_j, b_i, c, d \in C(\mathbb{R}^N \times [0, T]) \cap L^\infty(\mathbb{R}^N \times (0, T)),\]
\((L_I.1)\)
\[a^i_j, b_i \in L^\infty(0, T; W^{1,\infty}_{loc}(\mathbb{R}^N)),\]
\[d \leq 0.\]
$H_I \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N; \mathbb{R})$ is a nonlinear first order operator such that

\begin{equation}
H_I(\cdot,\cdot,0,0,0) \in L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^N)),
\end{equation}

that is that $H_I(\cdot,\cdot,0,0,0)$ plays the role of a source term and has the same rate of growth of the initial data.

Let us suppose that for each $R > 0$ there exists a modulus of continuity $\omega_R$ and a constant $l_R > 0$ such that

\begin{equation}
|H_I(x,t,0,0,0)| \leq h e^{m\|x\|} \text{ for } m > 0.
\end{equation}

Furthermore, we suppose that

\begin{equation}
M \in C(\mathbb{R}^2), \quad M(a,a) \equiv 0,
\end{equation}

\begin{equation}
M(a,b) - M(c,d) \leq M((a-c)^+ + |b-d|].
\end{equation}

We recall that assumptions (H$_T$.3)-(H$_T$.4) guarantees that Lemma 3.2.4 holds for the class of growth $g(\|x\|) = e^{m\|x\|}$, so that we have

\begin{equation*}
\mathcal{USC} \cap \bigcup_{n\leq m} L^\infty(0,T; L^\infty_{e^{m\|\cdot\|}}(\mathbb{R}^N)) \subseteq \mathcal{USC}^T,
\end{equation*}

\begin{equation*}
\mathcal{LSC} \cap \bigcup_{n\leq m} L^\infty(0,T; L^\infty_{e^{m\|\cdot\|}}(\mathbb{R}^N)) \subseteq \mathcal{LSC}^T
\end{equation*}

If these assumptions hold, then a comparison result for exponential growing solution can be derived.
Theorem 3.2.10 [5, Theorem 4.15] Let $u_0 \in C(\mathbb{R}^n) \cap L^\infty_{\exp}(\mathbb{R}^n)$ satisfies

\[ |u_0(x)| \leq B_0 e^{n_0\|x\|}, \quad n_0 < m. \]

Let $u \in USC \cap \bigcup_{n < m} L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^N))$ and $\pi LSC \cap \bigcup_{n < m} L^\infty(0,T; L^\infty_{\exp}(\mathbb{R}^N))$ be respectively subsolution and supersolution of the integro-differential Cauchy problem (3.2.4)-(3.2.2). Then

\[ u \leq \bar{u}, \quad \text{pointwise on } \mathbb{R}^n \times [0,T). \]

Financial application. As we have often mentioned, for what concerns the problem related to the pricing of an European derivative, the initial datum belongs to $C(\mathbb{R}^n) \cap L^\infty_{\exp}(\mathbb{R}^n)$. In force of Proposition 3.2.9 and of the comparison result 3.2.10, we can then give the following result:

Theorem 3.2.11 Under the assumptions $(L_{\mathcal{I},1})-(H_{\mathcal{I},1})$ the semilinear integro-differential Cauchy problem (3.2.4)-(3.2.2) is well posed in the class $C(\mathbb{R}^n) \cap \bigcup_{n < m} L^\infty_{\exp}(\mathbb{R}^n)$ in the framework of viscosity solutions, that is for all initial data $u_0 \in C(\mathbb{R}^n) \cap \bigcup_{n < m} L^\infty_{\exp}(\mathbb{R}^n)$ there exists a viscosity solution $u$ such that

\[ |u(x,t)| \leq ce^{n\|x\|}, \]

with

\[ n = \min \{ n' \in \mathbb{N} : e^{-n'\|x\|}u_0(x), e^{-n'\|x\|}H(x,t,0,0,0) \text{ are bounded} \} < m. \]

The solution obtained is unique in the class $L^\infty(0,T; \bigcup_{n < m} L^\infty_{\exp}(\mathbb{R}^N))$.

Coming back to the examples of Section 1.2.3, in particular to Example 1.2.10, the following result holds:

Corollary 3.2.12 [5, Proposition 4.17] Assume that the parameters $\alpha, \beta, \gamma, r$ satisfy some proper regularity conditions of continuity and Lipschitz continuity and the market is without arbitrage opportunities; then, for every final value $G, G \in C((0, +\infty)^N) \cap W^{1,\infty}_{\text{pol}}((0, +\infty)^N)$, the integro-differential Cauchy problem (1.2.16) has an unique viscosity solution $V$, in the sense of Definition 3.2.5, which belongs to $L^\infty(0,T; W^{1,\infty}_{\text{pol}}((0, +\infty)^N))$. Moreover, the comparison principle applies and we have that $V \geq 0$ whenever $G \geq 0$. 

Chapter 4

Numerical backgrounds

In this chapter we shall present some numerical backgrounds for finite difference schemes. The main reference for this chapter is [61] from which most of the contents are taken. To introduce basic notions, we consider linear partial differential equations of the form

\[ P(\partial_t, \partial_x)u = 0 \]  

and

\[ P(\partial_t, \partial_x, \partial_{xx})u = 0 \]

which are of first order in the derivative with respect to \( t \).

The prototype for partial differential equations of the first type is the wave equation

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \]  

where \( a \) is a real constant, while for the second one is the heat equation

\[ \frac{\partial u}{\partial t} = b \frac{\partial^2 u}{\partial x^2}, \]  

with \( b > 0 \) constant. We shall consider initial value problems and we discuss as example the numerical approximation for the Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, & x \in \Omega, t \in [0, T] \\
u(x, 0) = u_0(x), & x \in \Omega.
\end{cases}
\]  

If \( \Omega \) is bounded we have to assign boundary conditions.
4.1 Finite Difference Schemes for Linear Partial Differential Equations

The basic idea of finite difference schemes is to replace derivatives by finite differences. This can be done in many ways; as example we have

\[
\frac{\partial u}{\partial t}(mh, n) \approx \frac{u(mh, (n+1)k) - u(mh, nk)}{k} \approx u(mh, (n+1)k) - u(mh, (n-1)k). 
\]

That these are valid approximations is seen from the formulas

\[
\frac{\partial u}{\partial t}(x, t) = \lim_{\epsilon \to 0} \frac{u(x, t + \epsilon) - u(x, t)}{\epsilon} = \lim_{\epsilon \to 0} \frac{u(x, t + \epsilon) - u(x, t - \epsilon)}{2\epsilon}.
\]

Similar formulas approximate the derivatives respect to \(x\).

**Example 4.1.1** Let us consider as simple example the Cauchy problem (4.0.5). Discretizing first the space derivatives we obtain a set of ordinary differential equations in time that approximate the PDE. This procedure is called method of lines. At each point \(x_m\) we denote by \(u_m(t)\) a numerical approximation of \(u(x_m, t)\). First derivative can be approximate to second order accuracy, by

\[
\frac{du(x)}{dx} = \frac{u(x + h) - u(x - h)}{2h} - \frac{h^2}{6} \frac{d^3u}{dx^3}(\xi), \quad \xi \in [x - h, x + h]. \tag{4.1.1}
\]

This relation is valid for any smooth function \(u(x)\); by the Taylor expansion, we have

\[
u(x \pm h) = u(x) \pm h \frac{du(x)}{dx} + \frac{h^2}{2} \frac{d^2u(x)}{dx^2} + \frac{h^2}{6} \frac{d^3u}{dx^3}(\xi), \quad \xi \in [x - h, x + h], \tag{4.1.2}
\]

that gives (4.1.1).

Using (4.1.1) at points \(x_m\), we have

\[
\frac{\partial u}{\partial x}|_{(x_m, t)} = \frac{u(x_m + h, t) - u(x_m - h, t)}{2h} - \frac{h^2}{6} \frac{\partial^3u}{\partial x^3}(\xi_m, t),
\]

and neglecting second order terms in \(h\), we have the following set of ordinary differential equations

\[
\begin{aligned}
\frac{du_m(t)}{dt} &= -\frac{a}{2h} (u_{m+1}(t) - u_{m-1}(t)), \quad m = 1, \ldots, N \\
u_m(0) &= u_0(x_m), \quad m = 1, \ldots, N.
\end{aligned} \tag{4.1.3}
\]
Finite Difference Schemes for Linear Partial Differential Equations

This set of equations is supplemented with boundary conditions.

System (4.1.3) can be solved numerically by a scheme used for system of ordinary differential equations (ODE). If we use explicit Euler scheme, for example, we obtain the forward-time central-space approximation

\[ v_{m}^{n+1} = v_{m}^{n} - \frac{ak}{2h} (v_{m+1}^{n} - v_{m-1}^{n}). \]  

(4.1.4)

4.1.1 Convergence and Consistency

The most basic property that a scheme must have in order to be useful is that its solution approximate the solution of the corresponding PDE and that the approximation improves as the grid spacings, \( h \) and \( k \), tend to zero. A scheme that has such behavior is called a convergent scheme. To formally define these concepts, we consider the wide class of partial differential equations of the form defined in (4.0.1).

Definition 4.1.2 A one-step finite difference scheme approximating a partial differential equation is a convergent scheme if for any solution of the PDE, \( u(x, t) \), and solution to the finite difference scheme \( v_{m}^{n} \), such that \( v_{m}^{0} \) converges to \( u_{0}(x) \) as \( mh \) converges to \( x \), then \( v_{m}^{n} \) converges to \( u(x, t) \) as \( (mh, nk) \) converges to \( (x, t) \) as \( h, k \) tend to zero.

This definition is not complete until we clarify the nature of the convergence. The two most common discrete norm used to estimate the convergence error are the \( l^{\infty} \) norm

\[ \| v^{n} \|_{\infty, h} = \max \{ | v_{m}^{n} |, m \} , \]  

(4.1.5)

and the \( l^{2} \) norm

\[ \| v^{n} \|_{2, h} = \left( \sum_{m} | v_{m}^{n} |^{2} \right)^{1/2} . \]  

(4.1.6)

We will say that a scheme is convergent in norm \( \| \cdot \|_{\cdot, h} \) if

\[ \| v^{n} - u^{n} \|_{\cdot, h} \to 0, \]  

(4.1.7)

as \( k \to 0 \) and \( nk \to t \in (0, T) \), for all initial data \( u_{0} \) such that the corresponding initial value problem (4.0.1) is well-posed in the chosen norm \( \| \cdot \|_{\cdot, h} \).

Proving that a given scheme is convergent is not easy in general, if attempted in a direct manner. However, there are two related concepts that are easy to check, consistency and stability. First, we define consistency.

Definition 4.1.3 Given a partial differential equation \( Pu = 0 \) and a finite difference scheme \( P_{h,k}v = 0 \), we say that the finite difference scheme is consistent with the partial differential equation if for any smooth function \( \phi(x,t) \)

\[ P\phi - P_{h,k}\phi \to 0 \]  

as \( h, k \to 0 \),

the convergence being pointwise convergence at each grid point.
When we refer to a smooth function we mean one that is sufficiently differentiable for the context.

**Example 4.1.4** Let us consider again the simple example 4.1.1. For the wave equation (4.0.5), the operator $P$ is $\frac{\partial}{\partial t} + a \frac{\partial}{\partial x}$ so that

$$P\phi = \frac{\partial \phi}{\partial t} + a \frac{\partial \phi}{\partial x}, \quad \forall \phi \in C^{1,1}.$$ 

For the scheme (4.1.4), the difference operator $P_{h,k}$ is given by

$$P_{h,k}\phi = \frac{\phi^{n+1}_m - \phi^n_m}{k} + a \frac{\phi^n_{m+1} - \phi^n_{m-1}}{2h}.$$ 

As done in (4.1.2), the Taylor series of the function $\phi$ in $t$ and in $x$ about $(x_m, t_n)$ gives

$$\phi^{n+1}_m = \phi^n_m + k \phi_t + \frac{1}{2}k^2 \phi_{tt} + O(k^3),$$

$$\phi^n_{m\pm 1} = \phi^n_m \pm h \phi_x + \frac{1}{2}h^2 \phi_{xx} \pm O(h^3),$$

and so

$$P_{h,k}\phi = \phi_t + a \phi_x + \frac{1}{2}k \phi_{tt} + O(k^2) + O(h^2).$$

Thus

$$P\phi - P_{h,k}\phi = -\frac{1}{2}k \phi_{tt} + O(k^2) + O(h^2) \to 0 \text{ as } h, k \to 0.$$ 

Therefore, the scheme (4.1.4) is consistent and, as defined in (4.1.5), it is accurate of order $(1,2)$.

**Definition 4.1.5** A scheme $P_{h,k}v = 0$ that is consistent with the differential equation $Pu = 0$ is accurate of order $p$ in time and order $q$ in space if for any smooth function $\phi(x,t)$

$$P\phi - P_{h,k}\phi = O(h^q) + O(h^q).$$

We will say that this scheme is accurate of order $(p,q)$.

The quantity $P\phi - P_{h,k}\phi$ is called the truncation error of the scheme.

As we shall show later on, for some schemes the grid steps $k$ and $h$ cannot be chosen independently one from the other. Therefore, we give the following definition, which is more generally applicable. We assume that the time step $k$ is chosen as a function of the space step, i.e. $k = \Lambda(h)$, where $\Lambda$ is a smooth function of $h$ such that $\Lambda(0) = 0$.

**Definition 4.1.6** A scheme $P_{h,k}v = 0$ with $k = \Lambda(h)$ that is consistent with the differential equation $Pu = 0$ is accurate of order $r$ if for any smooth function $\phi(x,t)$

$$P\phi - P_{h,k}\phi = O(h^r).$$
Let us now fix our attention to the explicit difference schemes for problem (4.0.5). An explicit difference scheme is any scheme that can be written in the form

\[ v_{m}^{n+1} = \text{a finite sum of } v_{m'}^{n'} \text{ with } n' \leq n. \]

For explicit one-step scheme, we can write

\[ v_{m}^{n+1} = \sum_{j} c_{j,n} v_{j}^{n}. \]  (4.1.8)

For the particular case of explicit schemes of the form

\[ v_{m}^{n+1} = \alpha v_{m-1}^{n} + \beta v_{m}^{n} + \gamma v_{m-1}^{n-1}, \]  (4.1.9)

the following proposition holds.

**Proposition 4.1.7** Let \( k/h \) be fixed as \( k/h = \lambda \) with \( \lambda > 0 \) constant. The finite difference scheme (4.1.9) is consistent with the equation (4.0.5) if and only if \( \alpha + \beta + \gamma = 1 \) and \( \gamma - \alpha = \lambda a. \)

**Proof.** From Taylor expansion we have

\[ \phi_{m}^{n+1} = \phi_{m}^{n} + k\phi_t + \frac{1}{2}k^2\phi_{tt} + O(k^3), \]

\[ \phi_{m \pm 1}^{n} = \phi_{m}^{n} \pm h\phi_x + \frac{1}{2}h^2\phi_{xx} \pm \frac{1}{6}h^3\phi_{xxx} + O(h^4), \]

therefore,

\[ \frac{1}{k} \left[ \phi_{m}^{n+1} - (\alpha\phi_{m-1}^{n} + \beta\phi_{m}^{n} + \gamma\phi_{m-1}^{n}) \right] = \frac{1}{k} \left[ \phi_{m}^{n} + \phi_t + (\gamma - \alpha)\lambda^{-1}\phi_x + \frac{1}{2}k(\phi_{tt} - \lambda^{-2}(\alpha + \lambda)\phi_{xx}) \right] + O(k^2). \]

Then the thesis. \( \square \)

Consistency is certainly necessary for convergence, but as the following example shows, a scheme may be consistent but not convergent.

**Example 4.1.8** Consider the partial differential equation (4.0.5) with \( a = 1 \) and initial condition \( u_0(x) \) such that

\[ u_0(x) = \begin{cases} 
1 & \text{if } -1 \leq x \leq 0, \\
0 & \text{elsewhere}.
\end{cases} \]  (4.1.10)

The solution of this problem is a shift of \( u_0 \) to the right by \( t \). In particular, for \( t \) greater than zero, there are positive values of \( x \) for which \( u(x,t) \) is nonzero.

Consider the following forward-time forward-space scheme

\[ v_{m}^{n+1} = v_{m}^{n} - \frac{k}{h} \left( v_{m+1}^{n} - v_{m}^{n} \right). \]  (4.1.11)
This scheme is consistent and accurate of order $(1,1)$. Let $v_m^0$, for all $m \in \mathbb{Z}$, such that

$$v_m^0 = \begin{cases} 1 & \text{if } -1 \leq mh \leq 0, \\ 0 & \text{elsewhere} \end{cases}$$

As equation (4.1.11) shows the solution at $(t_n, x_m)$ depends only on $x_{m+1}$ at previous times. Thus,

$$v_m^n = 0, \quad \text{for } m > 0, n \geq 0.$$ 

Therefore, $v_m^n$ cannot converge to $u$, since for positive $t$ and $x$ the function $u$ is not identically zero, yet $v_m^n$ is zero. Notice that we conclude that the scheme is nonconvergent without specifying the type of convergence.

The same kind of argumentation may be used to prove that also the solution of the consistent forward-central scheme (4.1.4) cannot converge to the solution $u$ of (4.0.5)-(4.1.10) with $a > 0$.

### 4.1.2 Stability

Example (4.1.8) shows that a scheme must satisfy other conditions in addition to consistency in order to be convergent. The important property that is required is stability. To introduce this concept we note that if a scheme is convergent, then as $v_m^n$ converges to $u(x,t)$, certainly $v_m^n$ is bounded in some sense. This is the essence of stability.

**Definition 4.1.9** A finite difference scheme $P_h,kv_m^n = 0$ for a first order equation (4.0.1) is stable in the norm $\| \cdot \|_{.,h}$ in a stability region $R$ if for all $T > 0$ there is a constant $C_T$ such that, for all $v^0$,

$$\|v^n\|_{.,h} \leq C_T\|v^0\|_{.,h},$$

for $0 \leq nk \leq T$.

The importance of the concepts of consistency and stability is seen in the following theorem, the Lax-Richtmyer equivalence theorem, which is the fundamental theorem in the theory of finite difference schemes for initial value problems.

**Theorem 4.1.10 (Lax-Richtmyer)** A consistent finite difference scheme for a linear partial differential equation for which the initial value problem is well-posed is convergent if and only if it is stable.

Then, the class of "reasonable" schemes is delimited as those that are consistent and stable.

Before coming on the stability analysis, it is important to discuss the condition introduced on 1928 by R. Courant, K.O. Friedrichs and H. Lewy in their fundamental paper [22]. This condition is known as CFL condition and gives a necessary condition to convergence of a finite different scheme in terms of the dependency domain of the
Finite Difference Schemes for Linear Partial Differential Equations

Table 4.1: Three-points explicit finite difference schemes (4.1.14) for problem (4.0.5).

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$(\alpha, \beta, \gamma)$</th>
<th>$q$</th>
<th>Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upwind $(a &lt; 0)$</td>
<td>$(-\lambda a, (1 + \lambda a), 0)$</td>
<td>$-\lambda a$</td>
<td>$-1 \leq \lambda a &lt; 0$</td>
</tr>
<tr>
<td>Upwind $(a &gt; 0)$</td>
<td>$(0, (1 - \lambda a), \lambda a)$</td>
<td>$\lambda a$</td>
<td>$0 \leq \lambda a \leq 1$</td>
</tr>
<tr>
<td>Forward-Central</td>
<td>$(\lambda a/2, 1, \lambda a/2)$</td>
<td>$0$</td>
<td>Never</td>
</tr>
<tr>
<td>Lax-Friedrichs</td>
<td>$((1 - \lambda a)/2, 0, (1 + \lambda a)/2)$</td>
<td>$1$</td>
<td>$</td>
</tr>
<tr>
<td>Lax-Wendroff</td>
<td>$((\lambda a + \lambda^2 a^2)/2, 1 - \lambda^2 a^2, (-\lambda a + \lambda^2 a^2)/2)$</td>
<td>$(\lambda a)^2$</td>
<td>$</td>
</tr>
</tbody>
</table>

Theorem 4.1.11 For an explicit scheme for the hyperbolic equation (4.0.5) of the form (4.1.9) with $k/h = \lambda$ held constant, a necessary condition for stability is the Courant-Friedrichs-Lewy (CFL) condition,

$$|a\lambda| \leq 1, \text{ for } \lambda = k/h.$$  \hspace{1cm} (4.1.13)

Notice that this result can be extended to systems of hyperbolic equations. The unconditionally stability can be achieved for the class of implicit schemes.

Proof. Let be $a > 0$, considering the point $(x, t) = (0, 1)$ we see that the solution to the PDE (4.0.5) depends on the values of $u_0(x - at)$ at $x - at$. But, the three points finite difference scheme is such that $v^n_m$ depends on $v^0_m$ only for $|m| \leq n$. Let be $|a\lambda| > 1$. For $h = \lambda^{-1}k$ and $kn = 1$, we have $mh \leq \lambda^{-1}kn = \lambda^{-1}$. So $v^n_0$ can not converge to $u(0, 1)$ for $h \rightarrow 0$, in fact if we modify arbitrarily the initial data, the value $u(0, 1)$ modifies while $v^n_0$ not.

To conclude this part, we briefly point out that the class of linear explicit schemes consistent for problem (4.0.5) forms a family depending on a parameter. If we set

$$q = \alpha + \gamma,$$

we can write

$$v^{n+1}_m = v^n_m - \frac{a\lambda}{2}(v^n_{m+1} - v^n_{m-1}) + \frac{q}{2}(v^n_{m+1} - 2v^n_m + v^n_{m-1}).$$  \hspace{1cm} (4.1.14)

The $q$ parameter is connected with the numerical viscosity of the scheme. In Table 4.1 we show possibles coefficients choice. Notice that Lax-Wendroff scheme, which corresponds to the choice $q = (\lambda a)^2$, is second order accurate.
4.1.3 $l^\infty$-Stability and Monotonicity

A useful way to get $l^\infty$-stable approximations, is to construct schemes which verify the monotone comparison property, i.e if $v^0_m$ and $\tilde{v}^0_m$ are two approximations of the initial data such that $v^0_m \leq \tilde{v}^0_m$, then $v^n_m \leq \tilde{v}^n_m$ for all $n > 0$.

**Definition 4.1.12** A one-step scheme in the form (4.1.8), is monotone if $c_{j,n} \geq 0$ for all $j,n$.

**Theorem 4.1.13** Let be given a monotone scheme in the form (4.1.8) and note

$$v_{\min} := \min\{v^0_m\}, \quad v_{\max} := \max\{v^0_m\},$$

then, for all $n,m$

$$v_{\min} \leq v^n_m \leq v_{\max}. \quad (4.1.15)$$

Then, Theorem 4.1.10 allows us to state that all consistent and monotone schemes in the form (4.1.8), are convergent.

Unfortunately, the class of monotone schemes is too limited to hold more accurate approximations. Then, it is useful to analyze the class of $l^2$-stable schemes, which will turn out to be less restricted. In the next section, we shall give a tool to investigate the stability in norm $l^2$ of finite differences approximations to linear problems.

4.2 Fourier Analysis of Finite Difference Schemes

In this section we present the important tool of Fourier analysis, we shall use it to analyze finite difference schemes.

4.2.1 Fourier Analysis

Fourier analysis is the most extensively used tool to study stability of finite difference schemes for linear problems. We will use Fourier analysis in particular on the grid of integers, $\mathbb{Z}$ or $h\mathbb{Z}$, which is defined by $h\mathbb{Z} = \{hm : m \in \mathbb{Z}\}$.

Let $v = \{v_m\}$ be a grid function defined for all integers $m$. If the spacing between the grid points is $h$, its Fourier transform is given by

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} he^{-imh\xi}v_m, \quad (4.2.1)$$

for $\xi \in [-\pi/h, \pi/h]$ and the inversion formula is

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}(\xi)d\xi. \quad (4.2.2)$$
An important consequence of the preceding definitions is the Parseval’s relation. Let \( v \) be a bounded function in the norm \( l^2 \) defined in (4.1.6), so \( \hat{v} \in L^2 \) and the following relation holds

\[
\|\hat{v}\|_h^2 = \int_{-\pi/h}^{\pi/h} |\hat{v}(\xi)|^2 d\xi = \sum_{m=-\infty}^{+\infty} h |v_m|^2 = \|v\|_h^2. \tag{4.2.3}
\]

This relation allows us to replace stability estimate (4.1.12) by the equivalent inequality

\[
\|\hat{v}^n\|_{.h} \leq C^* T \|\hat{v}^0\|_{.h}, \tag{4.2.4}
\]

for the transform of the grid function. In the next subsection we study the stability of schemes by examining the effect of the schemes on the transform of the solution.

It should be useful to remember some properties of the Fourier transform used to study differential problems. Let for example \( u \) be the solution of problem (4.0.5) and \( \hat{u} \) its Fourier transform in the space variable, i.e.

\[
\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega x} u(x) dx.
\]

The function \( \hat{u} \) then solves the ODE

\[
\hat{u}_t = -i\omega \hat{u},
\]

that it has solution

\[
\hat{u}(\omega, t) = e^{-i\omega t} \hat{u}(\omega).
\]

### 4.2.2 The Von Neumann stability analysis

We illustrate the method by considering our particular example (4.0.5) with \( a > 0 \) and then discuss the method in general. Thought the use of the Fourier analysis the determination of the stability of a schemes is reduce to relatively simple algebraic considerations. We begin by considering the forward-time backward-space scheme

\[
v_m^{n+1} = (1 - \lambda a)v_m^n + \lambda av_{m-1}^n, \tag{4.2.5}
\]

where \( \lambda = k/h \). Using the Fourier inversion formula (4.2.2) for \( v^n \), we have

\[
v_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}(\xi) d\xi
\]

and substituting this in (4.2.5) for \( v_m^n \) and \( v_m^{n-1} \), we obtain

\[
v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \left[ (1 - \lambda a) + \lambda a e^{-ih\xi} \right] \hat{v}(\xi) d\xi. \tag{4.2.6}
\]
Comparing this formula with the Fourier inversion formula

\[ v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}^{n+1}(\xi) d\xi \]

and using the fact that the Fourier transform is unique, we deduce that the integrand of (4.2.6) is the same as that in the inversion formula. We then have that

\[ \hat{v}^{n+1}(\xi) = \left[ (1 - \lambda a) + \lambda e^{-ih\xi} \right] \hat{v}^n(\xi) \]

(4.2.7)

where

\[ g(h\xi) = (1 - \lambda a) + \lambda e^{-ih\xi}. \]

The formula (4.2.7) shows that advancing the solution of the scheme by one time step is equivalent to multiplying the Fourier transform of the solution by the amplification factor \( g(h\xi) \). From (4.2.7) we obtain the important formula

\[ \hat{v}^{n+1}(\xi) = g(h\xi)^n \hat{v}^0(\xi). \]

(4.2.8)

Note that the superscript on \( \hat{v} \) is an index of the time level, while on \( g \) is a power. We now use formula (4.2.8) to study the stability of scheme (4.2.5). By the Parseval’s relation, we have

\[ \sum_{m=-\infty}^{+\infty} h \left| v_m^n \right|^2 = \int_{-\pi/h}^{\pi/h} \left| \hat{v}^n(\xi) \right|^2 d\xi \]

\[ = \int_{-\pi/h}^{\pi/h} \left| g(h\xi)^n \hat{v}^0(\xi) \right|^2 d\xi. \]

To get stability in the norm \( L^2 \) we then need \( | g(h\xi) |^{2n} \) suitably bounded. We now evaluate \( | g(h\xi) | \). Setting \( \theta = h\xi \), we have

\[ | g(h\xi) |^2 = | (1 - \lambda a) + \lambda e^{-ih\xi} |^2 \]

\[ = \left( 1 - 2\lambda a \sin^2(\theta/2) \right)^2 + 4\lambda^2 a^2 \sin^2(\theta/2) \cos^2(\theta/2) \]

\[ = 1 - 4\lambda(1 - a\lambda) \sin^2(\theta/2). \]

Thus we see that \( | g(h\xi) | \) is bounded by 1 if \( 0 \leq a\lambda \leq 1 \), that is the result given in (4.1.13).

The exact condition for stability of constant coefficient one step schemes is given in the next theorem.
Theorem 4.2.1 A one-step finite difference scheme (with constant coefficients) is stable if and only if there is a constant $K$ (independent of $\theta$, $k$ and $h$) and some positive grid spacings $k_0$ and $h_0$ such that

$$|g(\theta; h, k)| \leq 1 + Kk$$

(4.2.9)

for all $\theta$, $0 < k \leq k_0$, and $0 < h \leq h_0$. If $g(\theta; h, k)$ is independent of $h$ and $k$, the stability condition (4.2.9) can be replaced with

$$|g(\theta)| \leq 1.$$  

(4.2.10)

This theorem shows that to determine the stability of a finite difference scheme we need to consider only the amplification factor $g(h\xi)$. This observation is due to von Neumann, and because of that, this analysis is usually called von Neumann analysis.

4.2.3 Symbol Analysis - Order of Accuracy of Finite Difference Schemes

In the previous two sections we classified schemes only on the basis of whether or not they are convergent, considering via the Lax-Richtmyer equivalence theorem, stability and consistency. However, two convergent schemes may differ considerably in how well their solution approximates the solution of the differential equation. In (4.1.5) and (4.1.6) we have defined the order of accuracy of a scheme, which can be regarded as an extension of the definition of consistency. We also have showed how to use the Taylor series to check for the accuracy of a scheme. In this subsection we shall show another useful way of checking for the accuracy of a scheme, by comparing the symbol of the difference schemes to the symbol of the differential operator.

Definition 4.2.2 The symbol $p_{h,k}(\xi, s)$ of a difference operator $P_{h,k}$ is defined by

$$P_{h,k}(e^{imh\xi}e^{skn}) = p_{h,k}(\xi, s)e^{imh\xi}e^{skn}.$$  

That is, the symbol is the quantity multiplying the grid function $e^{imh\xi}e^{skn}$ after operating on this function with the difference operator.

As an example, for the (4.1.4) operator we have

$$p_{h,k}(\xi, s) = e^{sk} - 1 - \frac{ia}{h}\sin(h\xi).$$

Definition 4.2.3 The symbol $p(\xi, s)$ of the differential operator $P$ is defined by

$$P(e^{ix\xi}e^{st}) = p(\xi, s)e^{ix\xi}e^{st}.$$  

That is, the symbol is the quantity multiplying the function $e^{ix\xi}e^{st}$ after operating on this function with the differential operator.
For (4.0.3) we have
\[ p(\xi, s) = s + ia\xi. \]

To give the theorem that shows the way of checking for the accuracy of a scheme for homogeneous equations, by comparing the symbols of the difference schemes to the symbol of the differential operator, we first need to give the following definition.

**Definition 4.2.4** A symbol \( q(\xi, s) \) is congruent to zero modulo a symbol \( p(\xi, s) \), written
\[ q(\xi, s) \equiv 0 \mod p(\xi, s), \]
if there is a symbol \( r(\xi, s) \) such that
\[ q(\xi, s) = r(\xi, s)p(\xi, s). \]

We can now enunciate the following theorem.

**Theorem 4.2.5** A scheme \( P_{h,k}v = 0 \), with \( k = \Lambda(h) \), that is consistent with the equation \( Pu = 0 \) is accurate of order \( r \) if
\[ p_{h,k}(\xi, s) \equiv O(h^r) \mod p(\xi, s). \] (4.2.11)

### 4.3 The \( \theta \)-schemes and the Crank-Nicolson scheme

In this section we shall consider the second order operator (4.0.2) and the heat equation (4.0.4). Well known examples approximating the heat equation are:

- **The Standard Explicit Scheme**
\[ v_{m+1}^n = v_m^n + \frac{bk}{h^2} \left( v_{m+1}^{n-1} - 2v_m^n + v_{m-1}^n \right), \] (4.3.1)

- **The Standard Implicit Scheme**
\[ v_{m+1}^n = v_m^n + \frac{bk}{h^2} \left( v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1} \right), \] (4.3.2)

Of course the definitions of the different schemes for \( m = 1 \) and \( m = M \) depends on the boundary condition we imposed.

It is easy to show that both explicit and implicit approximations are convergent schemes for (4.0.2); the second one in unconditionally stable while the first one is stable under the CFL condition
\[ \frac{2bk}{h^2} \leq 1, \] (4.3.3)

which means that when the spatial accuracy is increased by reducing \( h \) in half, then \( k \) must be reduced by one-fourth. This restriction on \( k \) can be quite severe for
practical computation. Moreover, by a Taylor expansion it is also easy to check that both schemes are accurate of order $(1,2)$, first order in time and second order in space. It is then useful to introduce other schemes to have higher accuracy.

A natural generalization of the two schemes (4.3.1), (4.3.2) is the following

\[ v_{m}^{n+1} = v_{m}^{n} + \frac{k}{h^{2}} \left[ \theta \left( v_{m}^{n+1} - 2v_{m}^{n+1} + v_{m-1}^{n+1} \right) + (1 - \theta) \left( v_{m}^{n} - 2v_{m}^{n} + v_{m-1}^{n} \right) \right], \]  

(4.3.4)

where $0 \leq \theta \leq 1$. It is clear that for $\theta = 0$ we get the explicit scheme (4.3.1), while for $\theta = 1$ the implicit one (4.3.2). Let be $\theta \neq 0$, by the von Neumann analysis we obtain

\[ g = \frac{1 - 4(1 - \theta)\mu \sin^{2}(\chi/2))}{1 + 4\theta\mu \sin^{2}(\chi/2)}, \]

for $\mu = k/h^{2}$. We have $g \leq 1$, we may then have instability for $g < -1$, i.e. if

\[ 4(1 - 2\theta)\mu \sin^{2}(\chi/2) > 2. \]

Therefor, for $0 \leq \theta < 1/2$ the method is $l^{2}$-stable if and only if

\[ \mu \leq \frac{1}{2} (1 - 2\theta)^{-1}, \]

while, for $1/2 \leq \theta \leq 1$ the method is $l^{2}$-stable for all $\mu$.

We then have to study the accuracy order of the scheme to choose which one is better to use. We note that to study the accuracy order by the Taylor series, it is very important the start point of the expansion. For explicit schemes is always convenient to start from $(x_{m},t_{n})$, while in this case, for scheme (4.3.4), it is convenient to start from $(x_{m},t_{n+\frac{1}{2}})$. Therefore, the truncation error is given by

\[ T_{m}^{n+\frac{1}{2}} = (\phi_{t} - \phi_{xx}) + \left[ \left( \frac{1}{2} - \theta \right) k\phi_{xxt} - \frac{1}{12} h^{2} \phi_{xxxx} \right] \]

\[ + \left[ \frac{1}{24} k^{3} \phi_{ttt} - \frac{1}{8} k^{2} \phi_{xxtt} \right] \]

\[ + \left[ \frac{1}{12} (1 - \theta) k h^{2} \phi_{xxxxxt} - \frac{2}{6!} h^{4} \phi_{xxxxxxx} \right]. \]

The scheme is then accurate of order $(1,2)$ for all $\theta \neq 1/2$. For $\theta = 1/2$ the scheme is accurate of order $(2,2)$ and it is called the Crank-Nicolson scheme. We notice that the Crank-Nicolson scheme is unconditionally stable.

### 4.4 The convection-diffusion equation

We now consider finite difference schemes for the convection-diffusion equation

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = b \frac{\partial^{2} u}{\partial x^{2}}, \]  

(4.4.1)

where $a$ and $b > 0$ are real constants.
Remark 4.4.1 Let \( y = x - at \) and set \( w(y, t) = u(y + at, t) \). Then,
\[
w_t = bw_{xx}.
\]
(4.4.2)
Since \( u(x, t) = w(x - at, t) \), the solution of (4.4.1), when examined from coordinate system moving with speed \( a \), is given by (4.4.2). Thus the solution of (4.4.1) travels with speed \( a \) (convection) and is dissipative with strength \( b \) (diffusion).

Remark 4.4.2 Notice that one property of parabolic differential equation (4.4.1) is that the maximum value of \( |u(x, t)| \) does not increase as \( t \) increases, i.e.
\[
\sup_x |u(x, t)| \leq \sup_x |u(x, t')|, \text{ if } t > t'.
\]
Then, it is natural to ask to its numerical approximations to satisfy a similar property, that is
\[
\max_m |v_{m+1}^n| \leq \max_m |v_m^n|.
\]
(4.4.3)

We study the forward-time scheme
\[
\frac{v_{m+1}^n - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = \left( \frac{q}{2k} + \frac{b}{h^2} \right) (v_{m+1}^n - 2v_m^n + v_{m-1}^n),
\]
(4.4.4)
where \( q \) is the “artificial viscosity” parameter defined in (4.1.14) and in Table 4.1. From arguments given in the previous sections, this scheme it is easy shown to be first-order accurate in time. The order accuracy in space depends on \( q \), in particular for the central-space and the Lax-Wendroff scheme, \( q = 0 \) and \( q = (a\lambda)^2 \) respectively, the scheme is second order in space.

Scheme (4.4.4) may be written in the equivalent form
\[
v_{m+1}^n = \left( \frac{\beta}{2} + \frac{\alpha}{2} \right) v_{m-1}^n + (1 - \beta) v_m^n + \left( \frac{\beta}{2} - \frac{\alpha}{2} \right) v_{m+1}^n,
\]
(4.4.5)
where
\[
\beta = q + \frac{2bk}{h^2}, \quad \alpha = a\lambda, \quad \lambda = \frac{k}{h}.
\]
We assume \( a > 0 \), the case when \( a < 0 \) is similar. From (4.4.5), we see that the scheme is \( l^\infty \)-stable if and only if
\[
\alpha \leq \beta \leq 1.
\]
(4.4.6)
That is, if \( \alpha \) and \( \beta \) satisfy (4.4.6), then we have
\[
|v_{m+1}^n| \leq \left( \frac{\beta}{2} + \frac{\alpha}{2} \right) |v_{m-1}^n| + (1 - \beta) |v_m^n| + \left( \frac{\beta}{2} - \frac{\alpha}{2} \right) |v_{m+1}^n| \leq \max_m |v_m^n|,
\]
and thus
\[ \|v^n\|_{\infty,h} \leq \|v^0\|_{\infty,h}. \]

As it were natural to expect, we get \( \alpha \leq 1 \), that is the hyperbolic stability condition (4.1.13). The condition \( \alpha \leq \beta \leq 1 \) depends on \( q \). Notice that scheme (4.4.4) is consistent for \( q \to 0 \) and condition (4.4.6) gives
\[ \frac{ak}{h} - \frac{2bk}{h^2} \leq q \leq 1 - \frac{2bk}{h^2}. \]

Then, at once the choice \( q = 1 \) is unsuitable. When \( q = 0 \) the scheme is central in space and forward in time. The stability condition (4.4.6) becomes,
\[ \frac{2bk}{h^2} \leq 1, \]
\[ h \leq \frac{2b}{a}. \]

(4.4.7)

We recognize the first one as the parabolic stability condition (4.3.3), while the second one needs some remarks. It does not look like a stability relation, since stability only deals with the limit as \( h \) and \( k \) tend to zero, (4.4.7) is a restriction only on the spatial mesh spacing \( h \) and it is always satisfied for \( h \) small enough. Therefore, the oscillations that occur when (4.4.7) is violated are not the result of instability, they are only the result of inadequate resolution. In fact, it is easy to check, that if (4.4.7) is violated, then the scheme does not satisfy the maximum property (4.4.3) and its solution cannot converge to the exact solution of (4.4.1).

**Remark 4.4.3** The quantity \( a/b \) corresponds to the Reynolds number in fluid dynamic or to the Peclet number in heat flow and the quantity \( ha/2b \), or \( ha/b \) is often called the “cell Reynolds number” or “cell Peclet number” of the scheme.

One common way to avoid the restriction (4.4.7) is to use *upwind differencing* of the convection term, that is \( q = \alpha \) in (4.4.4). The condition (4.4.7) has been eliminated at the expense of being only first-order accurate in space. The stability condition is now given by
\[ \alpha + \beta = \frac{ak}{h} + \frac{2bk}{h^2} \leq 1. \]

When \( b \) is small and \( a \) is large (the convection term is dominant), this condition is less restrictive than (4.4.7).

For \( q = (a\lambda)^2 \) the condition (4.4.6) becomes,
\[ a\lambda \leq (a\lambda)^2 + \frac{2b\lambda}{h} \leq 1. \]

In every case, the parabolic condition (4.3.3), that states \( k = O(h^2) \), is required, then the scheme is globally second-order accurate.

To get high order approximations in time, a common way is to apply the method
of lines to the differential equation obtaining a set of ordinary differential equations in time and then to use a high order scheme for the ODE system, as for instance Runge-Kutta schemes, which we shall introduce in the next section.

Moreover, to avoid the heavy condition (4.3.3), it is needed to involve implicit mechanisms in the scheme. To generalize the scheme (4.4.4) to a scheme which is implicit in time, it is natural to apply the $\theta$-technique used in (4.3.4), as the Crank-Nicolson scheme. This allows to avoid both condition (4.3.3) and condition (4.1.13).

However, when for example, we deal with problems that involves nonlinear first-order terms, it is unseemly to solve them implicitly. Therefore, it is natural idea to employ different discretizations to different terms of the equation. Implicit-Explicit (IMEX) time-discretization schemes [43, 17], are an example of such a strategy. Typically, for convection-diffusion equations, an implicit scheme is used for the diffusion term and an explicit scheme is used for the convection term. This allows to avoid heavy conditions, such as (4.3.3) and to raise time orders accuracy.

Some schemes of this type were proposed and analyzed as far back as the late 1970’s. Instances of this methods have been successfully applied to the incompressible Navier-Stokes equations [41] and in environmental modeling studies [62]. We refer to [54, 52, 53] for recent developments. We shall investigate these schemes in the last Chapter of this Thesis, as effectives approximations of integro-differential equations.

### 4.5 Runge-Kutta schemes

Let us consider Example 4.1.1. As we have mentioned discretizing first the space derivative we obtain a set of ordinary differential equations in time of the form

\[
\begin{align*}
\frac{dv(t)}{dt} &= F(t,v), \\
v(t_0) &= v_0,
\end{align*}
\]

where $F$ depends on the space discretization. System (4.5.1) can be solved numerically by a scheme used for ODE, as Runge-Kutta scheme.

A Runge-Kutta scheme of $\nu$-levels is obtained as follows: given $v^n \approx v(t_n)$, compute $v^{n+1} \approx v(t_{n+1})$ as

\[
v^{n+1} = v^n + k \sum_{i=1}^{\nu} \omega_i r_i
\]

\[
r_i = F(t_n + c_i k; v^n + \sum_{j=1}^{\nu} a_{ij} r_j), \quad i = 1, \ldots, \nu.
\]

Rung-Kutta schemes are characterized by the $\nu \times \nu$ real matrix $A = (a_{ij})$ and by the vectors $c = (c_1, \ldots, c_\nu)^T$ and $\omega = (\omega_1, \ldots, \omega_\nu)^T$. If the matrix $A$ is lower triangular,
with zero diagonal, then the method is explicit (the values $r_i$ can be “explicitly” computed from $r_j$, $1 \leq j \leq i - 1$). Runge-Kutta schemes are said one-step schemes, since they use only values of $v^n$ to compute $v^{n+1}$.

The concept of consistency, stability and convergence for ODE are the same of the ones given previously for PDE. We limit then here to recall the concept of a region of absolute stability of a Runge-Kutta scheme for ODE’s system.

In the study of the stability one usually considers a class of equations and verifies the stability properties of that class of equations. The study of Absolute Stability (or A-stability) concerns single scalar equation of the form

$$
\begin{align*}
\begin{cases}
  y' = \lambda y \\
y(0) = y_0,
\end{cases}
\end{align*}
$$

(4.5.3)

with $y(t) : \mathbb{R} \to \mathbb{C}$, $\lambda \in \mathbb{C}$, $\mathcal{R}\lambda \leq 0$. It is $y(t) = y_0 e^{\lambda t}$, which implies

$$
|y(t)| = \exp(\mathcal{R}\lambda t).
$$

Therefore if $\mathcal{R}\lambda \leq 0$ then $|y(t)|$ is a decreasing function. When applying the numerical method to this equation one wants to observe the qualitative behavior of the numerical solution with respect to growing and decreasing. Notice that the study of a single complex equation gives information on a linear system of the form

$$
\begin{align*}
\begin{cases}
  y' = Ay \\
y(0) = y_0,
\end{cases}
\end{align*}
$$

(4.5.4)

with $y_0 \in \mathbb{R}^m$, $y(t) : \mathbb{R} \to \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$, since the asymptotic behavior of the solution depends on the eigenvalues of the matrix $A$.

By applying Runge-Kutta scheme to equation (4.5.3) to compute $y_1 \approx y(t_1)$ one has

$$
y_1 = \left(1 + k \sum_{i=1}^{\nu} \omega_i r_i\right) y_0
$$

(4.5.5)

$$
r_i = \lambda \left(1 + \sum_{j=1}^{\nu} a_{ij} r_j\right), \quad i = 1, \ldots, \nu.
$$

Introducing the vector $e = (1, \ldots, 1)^T$, $r = (r_1, \ldots, r_{\nu})^T$, one can write (4.5.5) in the form

$$
y_1 = (1 + k \omega^T r) y_0
$$

(4.5.6)

$$
r = \lambda(e + k A r).
$$

Therefore,

$$
y_1 = (1 + \omega^T (I - \lambda k A)^{-1} k \lambda e) y_0.
$$
Let $z \equiv \lambda k$, and let $R(z) \equiv (1 + z\omega^T(I - zA)^{-1}e)$. Then one has

$$y_1 = R(z)y_0,$$

and in general,

$$y_n = R(z)^n y_0.$$ 

Therefore, if $|R(z)| < 1$ the numerical solution decreases, and if $|R(z)| > 1$ the numerical solution increases (of course in absolute value). The function $R(z)$ is called absolute stability function. The region of A-stability is the region of the complex plane where $|R(z)| \leq 1$. Formally

**Definition 4.5.1** The region

$$S_A \equiv \{ z \in \mathbb{C} : \exists (I - zA)^{-1} \text{ and } |R(z)| \leq 1 \}$$

is called the region of absolute stability of the Runge-Kutta method.

**Definition 4.5.2** A Runge-Kutta scheme is said A-stable if its region of absolute stability $S_A$ contains the complex half plane $\mathbb{C}^- = \{ z \in \mathbb{C} : Re z \leq 0 \}$. i.e. $S_A \supseteq \mathbb{C}^-$. 

**Example 4.5.3** Let us analyses the A-stability for the explicit and implicit Euler scheme and for the Midpoint method, that is a one-stage implicit Runge-Kutta scheme.

- Explicit Euler scheme is simply

$$y_{n+1} = y_n + kf(t_n, y_n),$$

then $R(z) = 1 + z$, and $S_A = \{ z \in \mathbb{C} : |1 + z| \leq 1 \}$, so $S_A$ is the region inside the circle centered in $(-1, 0)$ and radius 1. The scheme is of coarse not A-stable.

- Implicit Euler is given by

$$y_{n+1} = y_n + kf(t_{n+1}, y_{n+1}),$$

therefore

$$R(x) = \frac{1}{1 - z}, \quad S_A = \left\{ z \in \mathbb{C} : \frac{1}{1 - z} \leq 1 \right\},$$

so $S_A$ is the region outside the circle centered in $(1, 0)$ and radius 1. Since this region contains $\mathbb{C}^-$ this scheme is A-stable.

- Midpoint scheme is given by
Runge-Kutta schemes

\[ p = \nu \begin{array}{c|c}
1 & R(z) \\
2 & 1 + z + z^2/2 \\
3 & 1 + z + z^2/2 + z^3/6 \\
4 & 1 + z + z^2/2 + z^3/6 + z^4/24 \\
\end{array} \]

Table 4.2: Functions of absolute stability for explicit Runge-Kutta schemes.

\[ y_{n+1} = y_n + kr_1, \]

with

\[ r_1 = f\left(t_n, \frac{k}{2}, y_n + \frac{k}{2}r_1\right). \]

Applying it to the equation \( y' = \lambda y \) one has

\[ r + 1 = \lambda \left(1 + \frac{k}{2}r_1\right) \implies r_1 = \frac{\lambda}{1 - z/2} \]

and therefore

\[ R(z) = \frac{2 + z}{2 - z}. \]

The region of absolute stability is

\[ S_A = \{z \in \mathbb{C} : \frac{2 + z}{2 - z} \leq 1\} \]

which correspond to \( \mathbb{C}^- \). This scheme has the remarkable property \( S_A = \mathbb{C}^- \) and it it therefore A-stable.

We conclude this short part remarking some more facts about Runge-Kutta schemes. It is easy to show that the function of absolute stability on an explicit Runge-Kutta scheme is a polynomial in \( z \), therefore an explicit Runge-Kutta scheme can not be A-stable. Runge-Kutta schemes of order \( p \) can be determined by imposing that the local truncation error is an infinitesimal of order \( p + 1 \) in \( k \). Explicit Runge-Kutta schemes with \( \nu = p \) exist for \( p = 1, 2, 3, 4 \). For them, the function of absolute stability takes the form of a Taylor expansion of the exponential, as showed in Table 4.2.
Chapter 5

Numerical approximations: the purely Differential case

In this chapter we would like to give an overview of the convergence results for degenerate fully nonlinear second order equations, with particular care for problems arising in Finance Theory. Several work has been done in this field [11, 8, 9] in the viscosity solutions setting.

We will present a convergence result for the following problem

\[ \partial_t u + F(x, t, u, D u, D^2 u) = 0, \]

(5.0.1)

with an initial data

\[ u(x, t) = u_0(x); \]

(5.0.2)

where \( u_0 \in C(\mathbb{R}^n) \) and \( F \in C(\mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n) \). These kind of problem arises in Finance when pricing a derivative in a market driven by a continuous process, like in the Black and Scholes market [14]. Nonlinearity effects arises when considering a market with large investors (see Subsection 1.2.3 and references therein).

In [11] a convergence result for monotone stable and consistent scheme is given, provided that suitable assumptions on the pure differential problem holds. The convergence result relies on a passage to the limit in fully nonlinear second order partial differential equations based only on \( L^\infty \) estimates and on the notion of viscosity solutions.

Furthermore, starting from the notion of finite difference schemes given in Chapter 4, we shall discuss numerical approximations of the solution of the Black and Scholes equation (1.1.3).

For all the definition and results in viscosity solutions theory, we refer to Chapter 3 and [23].
5.1 Discontinuous viscosity solutions and numerical schemes.

Let us consider the following problem

\[ \partial_t u + F(x, t, u, Du, D^2 u) = 0, \text{ in } \mathbb{R}^n \times [0, T] \]  

(5.1.1)

where \( F : \mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times S_n \to \mathbb{R} \) and \( u \) are locally bounded, possibly discontinuous. The problem (5.1.1) is stated on the closed set \([0, T]\), as it has been done in [11] because we would like to write both the equation and the “boundary condition” in one expression: let define \( F \) as follows

\[
F = \begin{cases} 
F(x, t, u, Du, D^2 u), & \text{if } (x, t) \in \mathbb{R}^n \times (0, T], \\
 u - u_0 & \text{if } (x, t) \in \mathbb{R}^n \times \{0\},
\end{cases}
\]

from which derives the possibly discontinuity of the operator \( F \).

We assume that \( F \) is an elliptic operator; that is

\[ F(x, t, u, p, X) \leq F(x, t, u, p, Y), \text{ for all } X, Y \in S_n \text{ such that } X \geq Y. \]  

(5.1.2)

Because of the possible discontinuity of \( u \) and \( F \), we have to restate the definition of viscosity solution as follows.

**Definition 5.1.1** A locally bounded function \( u : \mathbb{R}^n \to \mathbb{R} \) is a viscosity supersolution (respectively subsolution) of (5.1.1) if for all \( \phi \in C^\infty(\mathbb{R}^n \times [0, T]) \) and all \( (x, t) \in \mathbb{R}^n \times [0, T] \) such that \( u^* - \phi \) (respectively, \( u_* - \phi \)) has a local maximum point (respectively, minimum) at \( (x, t) \) we have

\[ \overline{F}_*(x, t, u^*(x, t), D\phi(x, t), D^2\phi(x, t)) \leq 0, \]

(respectively

\[ \underline{F}^*(x, t, u_*(x, t), D\phi(x, t), D^2\phi(x, t)) \geq 0. \])

The function \( u \) is a viscosity solution if it is both a sub/supersolution of (5.1.1).

In this way the problem to study becomes

\[ \partial_t u + \overline{F}(x, t, u, Du, D^2 u) = 0, \text{ on } \mathbb{R}^n \times [0, T]. \]

A locally bounded function \( u \) is a viscosity solution of (5.1.1)-(5.0.2) in the sense of Definition 5.1.1 if it satisfies in viscosity sense the following

\[ \partial_t u + F(x, t, u, Du, D^2 u) = 0, \text{ in } \mathbb{R}^n \times (0, T], \]  

(5.1.3)

\[ \max \left\{ \partial_t u + F(x, t, u, Du, D^2 u), u - u_0 \right\} \geq 0, \text{ on } \mathbb{R}^n \times \{0\}, \]  

(5.1.4)
A general convergence result.

It has been noticed that the previous equation (5.1.3)–(5.1.4)–(5.1.5) seems to be the natural replacement of problem (5.1.1)–(5.1.2), which does not have in general a solution which assumes continuously the boundary condition.

5.2 A general convergence result.

Let us consider an approximation scheme of the form

\[ S(h, k, j, n, u^j_n, \tilde{u}) = 0, \text{ in } \mathbb{R}^n \times [0, T]; \]  \hspace{1cm} (5.2.1)

here \( S : \mathbb{R} \times \mathbb{R} \times \mathbb{Z}^+ \times \mathbb{N} \times \mathbb{R} \times B(\mathbb{R}^n \times [0, T]) \to \mathbb{R} \), where \( B(\mathbb{R}^n \times [0, T]) \) indicates the set of all bounded functions on \( \mathbb{R}^n \times [0, T] \).

It will be proved that as far as the proposed scheme is monotone, stable and consistent, its solution converges to the unique continuous viscosity solution of (5.1.1), provided a comparison principle for this equation has been proved.

Let us suppose the following assumptions on the scheme \( S \) hold:

**Properties of the Scheme.**

**S.1 Monotonicity of the approximation** If \( \tilde{u} \leq \tilde{v} \) and \( u^j_n = v^j_n \) for all \( h, k \) and \( 0 \leq n \leq N \) then:

\[ S(h, k, j, n, u^j_n, \tilde{u}) \leq S(h, k, j, n, v^j_n, \tilde{v}). \]  \hspace{1cm} (5.2.2)

**S.2 Stability**

For all \( h, k \) a solution \( \hat{u} \) does exist that is bounded independently from \( (h, k) \);  \hspace{1cm} (5.2.3)

**S.3 Consistency**

For all \( \phi \in \mathcal{C}_b^\infty(\mathbb{R}^n \times [0, T]) \) and for all \( (x, t) \in \mathbb{R}^n \times (0, T) \) we have:

\[ \liminf_{(h,k) \to 0} \limsup_{(j,h,nk) \to (x,t)} \lim_{\xi \to 0} \frac{S(h, k, j, n, \phi^j_n + \xi, \bar{\phi} + \xi)}{\rho(h, k)} \geq \partial_t u + F^*(x, t, u, Du, D^2u); \]  \hspace{1cm} (5.2.4)

\[ \limsup_{(h,k) \to 0} \liminf_{(j,h,nk) \to (t,x)} \lim_{\xi \to 0} \frac{S(h, k, j, n, \phi^j_n + \xi, \bar{\phi} + \xi)}{\rho(h, k)} \leq \partial_t u + F^*(x, t, u, Du, D^2u); \]  \hspace{1cm} (5.2.5)
Property of the Equation

**S.4 Maximum principle or Strong uniqueness property.** If \( u \in B(\mathbb{R}^n \times [0,T]) \) is an upper semicontinuous solution of (5.1.1) and \( v \in B(\mathbb{R}^n \times [0,T]) \) is a lower semicontinuous solution of (5.1.1), then
\[
u \leq v, \text{ on } \mathbb{R}^n \times [0,T].
\]

Under this theoretical assumptions the following convergence result holds.

**Theorem 5.2.1** [11, Theorem 2.1, page 275] Let assumptions (S.1)–(S.4) hold true. Then as \((h, k) \to 0\), the solution of the scheme (5.2.1) converges locally uniformly to the unique continuous viscosity solution of the problem (5.1.1)–(5.0.2).

**Remark 5.2.2** If the function \( F \) is continuous, the consistency requirement is equivalent to
\[
\frac{S(h, k, m, n, \phi + \xi, \hat{\phi} + \tilde{\xi})}{\rho(h, k)} \to G(x, t, \phi, \frac{\partial \phi}{\partial t}, D\phi, D^2\phi). \tag{5.2.6}
\]
when \( \rho(h, k) \to 0 \) uniformly, for any smooth function \( \phi \). We recover here a more standard formulation (see Definition 4.1.3) and the apparent complexity of the consistency assumption above just comes from the fact that we have considered the presence of boundary conditions which leads to a discontinuous function \( F \).

**Remark 5.2.3** The monotonicity assumption (5.2.2) is the discrete analogue of the ellipticity condition (5.1.2); it ensures some maximum principle type property for the scheme. It can be noticed that it can be relaxed in several ways as inequality (5.2.2) needs only to hold up to \( o(\rho(h, k)) \) terms.

**Proof.** Let \( \underline{u}, \overline{u} \in B(\mathbb{R}^n \times [0,T]) \) be defined respectively by
\[
\underline{u}(x, t) := \liminf_{(\Delta t, \Delta x) \to 0} \liminf_{(n \Delta t, n \Delta x) \to (t,x)} u^n_j, \tag{5.2.7}
\]
\[
\overline{u}(x, t) := \limsup_{(\Delta t, \Delta x) \to 0} \limsup_{(n \Delta t, n \Delta x) \to (t,x)} u^n_j. \tag{5.2.8}
\]

We would like to prove that \( \overline{u} \) and \( \underline{u} \) are respectively sub- and supersolutions of (5.1.1); if this claim is proved to be true, by definition we have \( \underline{u} \leq \overline{u} \), the other inequality holding because of lower semicontinuity of \( \underline{u} \) and upper semicontinuity \( \overline{u} \), therefore
\[
u = \underline{u} = \overline{u},
\]
is the unique continuous solution to (5.1.1). Local uniform convergence of the solution of the scheme to the solution of the problem is gained by (5.2.7)–(5.2.8) and the continuity of \( u \).

To prove the claim we shall consider only the case of \( \bar{\varphi} \), the other being the same. Let \( \phi \in C^\infty_b(\mathbb{R}^n \times [0,T]) \) and assume \((x_0,t_0)\) be a local maximum of \( \bar{\varphi} - \phi \) on \( \mathbb{R}^n \times [0,T] \); without loss of generality we can assume that the maximum point is a strict local maximum and \( u(x_0,t_0) = \phi(x_0,t_0) \); moreover we can assume that

\[
\phi \geq 2 \sup_{j \in \mathbb{Z}^+, n \in \mathbb{N}} \| u_n^j \|, \text{ outside the ball } B((x_0,t_0), r),
\]

where \( r > 0 \) is such that

\[
\bar{\varphi}(x,t) - \phi(x,t) \leq 0 = \bar{\varphi}(x_0,t_0) - \phi(x_0,t_0), \text{ in } B((x_0,t_0), r).
\]

From these assumptions it follows that there exist a sequence \((\Delta t_k, \Delta x_k) \in \mathbb{R}^+ \times (y_k, s_k) \in \mathbb{R}^n \times [0,T] \) such that, as \( k \to \infty \)

\[
(\Delta t_k, \Delta x_k) \to 0, \quad (y_k, s_k) \to (x_0, t_0), \quad u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) \to \bar{\varphi}(x_0,t_0),
\]

\[
(y_k, s_k) \text{ is a global maximum point of } u^{(\Delta t_k, \Delta x_k)}(\cdot, \cdot) - \phi(\cdot, \cdot). \quad (5.2.9)
\]

Denoting by \( \xi_k = u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) - \phi(y_k, s_k) \), we have

\[
\xi_k \to 0 \quad \text{and} \quad u^{(\Delta t_k, \Delta x_k)}(x,t) \leq \phi(x,t) + \xi_k,
\]

for all \((x,t) \in \mathbb{R}^n \times [0,T] \). By the definition of \( u^{(\Delta t_k, \Delta x_k)} \), the hypotheses (5.2.2), (5.2.9)

\[
S(\Delta t, \Delta x, n_k, j_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k) \leq 0.
\]

Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:

\[
0 \geq \liminf_{k \to \infty} \frac{S(\Delta t_k, \Delta x_k, n_k, j_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k)}{\rho_k(\Delta t, \Delta x)} \geq \liminf_{(\Delta t, \Delta x) \to 0} \liminf_{(n \Delta t, j \Delta x) \to (t,x)} \liminf_{\xi \to 0} \frac{S(\Delta t, \Delta x, n, j, \phi(y, s) + \xi, \tilde{\phi} + \xi)}{\rho(\Delta t, \Delta x)} \geq \partial_t \phi + F_s(x,t,\phi, D\phi, D^2\phi),
\]

which is the desired result, because of the assumption \( \bar{\varphi}(x_0,t_0) = \phi(x_0,t_0) \).
5.3 Financial applications

In this section we consider some application to the pricing problem, which is the main interest of this work.

The class of growth in which establish an existence and uniqueness result is of fundamental importance: solutions do exist in the class of growth determined by the initial (final) datum of the problem. Classical results have been established in the class of solution with linear growth at infinity, while the problem of financial interest have initial data which typically grow up exponentially at infinity.

The convergence result by Barles and Souganidis [11] has been established in the class of locally bounded solution, but it can be easily extended to the class of exponential growth at infinity, as it suffices to consider a suitable monotonicity result for the differential problem.

**Property of the equation**

S.4 **Maximum principle or Strong uniqueness property.** Let \( u_0 \in C(\mathbb{R}^n) \cap L^\infty_\text{exp}(\mathbb{R}^n) \) be the initial data of (5.1.1), (5.0.2), such that there exists \( m > 0 \):

\[
|u_0(x)| \leq B e^{n_0\|x\|} \quad \text{for} \quad n_0 \leq m;
\]

if \( u \in \bigcup_{n < m} L^\infty(0, T; L^\infty_{e^{n\|\cdot\|}}(\mathbb{R}^n)) \) is an upper semicontinuous subsolution of (5.1.1) and \( v \in \bigcup_{n < m} L^\infty(0, T; L^\infty_{e^{n\|\cdot\|}}(\mathbb{R}^n)) \) is a lower semicontinuous supersolution of (5.1.1), then

\[
u \leq u \quad \text{on} \quad \mathbb{R}^n \times [0, T].\]

With this property the general convergence result 5.2.1 still holds even in the financial setting.

**Theorem 5.3.1 [11, Theorem 2.1, page 275]** Let assumptions (S.1)–(S.4) hold true. Then as \( (h, k) \to 0 \), the solution of the scheme (5.2.1) converges locally uniformly to the unique continuous viscosity solution of the problem (5.1.1)–(5.0.2).

**Proof.** Let \( \underline{u}, \overline{u} \in \bigcup_{n < m} L^\infty(0, T; L^\infty_{e^{n\|\cdot\|}}(\mathbb{R}^D)) \) be defined by:

\[
\underline{u}(x, t) := \liminf_{(\Delta t, \Delta x) \to 0} u_j^n, \quad \text{if} \quad (n \Delta t, j \Delta x) \to (t, x);
\]

\[
\overline{u}(x, t) := \limsup_{(\Delta t, \Delta x) \to 0} u_j^n, \quad \text{if} \quad (n \Delta t, j \Delta x) \to (t, x);
\]

We want to prove that \( \underline{u} \) and \( \overline{u} \) are respectively supersolution and subsolution of the problem (5.1.1). If this claim is proved to be true, then by definition we have
\( u \leq \overline{u} \), while the other inequality holds because of lower semicontinuity of \( u \) and upper semicontinuity of \( \overline{u} \), hence:

\[
    u = u = \overline{u},
\]

is the unique continuous solution of the problem (5.1.1). This result, together with the definition of \( u \) and of \( \overline{u} \) leads to the local uniform convergence of the solution of the scheme to the solution of the problem. To prove the claim, we shall consider only the case of \( u \), the other being the same.

We want to prove that \( u \in USC \) and that it is a subsolution for the problem, i.e. for all \( \phi \in C^{1,1}(\mathbb{R}^n \times [0, T]) \) such that \( u - \phi \) has a local maximum in \((x_0, t_0)\) we have:

\[
    \partial_t \phi(x_0, t_0) + F(x_0, t_0, \overline{u}(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \leq 0.
\]

We start by proving that \( u \in USC \).

1. \( u \) is upper semicontinuous: it follows from the definition itself: we want to prove that the following inequality holds:

\[
    \limsup_{(y,s) \to (x,t)} u(y,s) \leq u(x,t),
\]

By definition:

\[
    \overline{u}(y,s) = \limsup_{(\Delta t, \Delta x) \to 0} \sup_{(n\Delta t,j\Delta x) \to (s,y)} u^n_j;
\]

therefore, by definition of \( \limsup \), there exist an \( \epsilon > 0 \) and \( (n,j) \) such that:

\[
    \overline{u}(y,s) - \epsilon \leq u^n_j;
\]

now, taking \( \limsup \) for \((\Delta t, \Delta x) \to 0 \) and \((n \Delta t, j \Delta x) \to (t,x)\), we obtain:

\[
    \overline{u}(y,s) - \epsilon \leq \overline{u}(x,t);
\]

Now, as \( \epsilon \) is arbitrarily chosen, we obtain the desired result.

2. \( u \) is locally bounded: by definition it is obtained by the solutions of the scheme. By hypothesis (5.2.3) they are bounded, independently from \( \Delta t, \Delta x \). So let \( K \in \mathbb{R}^N \times [0, T] \) be a compact set, then there exists a constant \( A_K \) such that:

\[
    |u^n_j| \leq A_K \quad \forall n, j \text{ s.t. } (n \Delta t, j \Delta x) \in K \quad \Rightarrow \quad |\overline{u}(x,t)| \leq A_K \quad \forall (x,t) \in K;
\]

We have proved that \( \overline{u} \in USC \), and now we need only to prove that \( \overline{u} \) is a viscosity subsolution. To this aim let \((x_0, t_0)\) be a global strict maximum for \( \overline{u} - \phi \) on \( \mathbb{R}^n \times [0, T] \) for some \( \phi \in C^\infty_b(\mathbb{R}^n \times [0, T]) \). We could assume that \( \overline{u}(x_0, t_0) = \phi(x_0, t_0) \) and that:

\[
    \overline{u}(x,t) - \phi(x,t) \leq 0 = \overline{u}(x_0, t_0) - \phi(x_0, t_0) \text{ in } \mathbb{R}^n \times [0, T].
\]
From these hypothesis it follows that there exists a sequence \((\Delta t_k, \Delta x_k) \in \mathbb{R}^+ \times [0, T]\) such that, as \(k \to \infty\):

\[
(\Delta t_k, \Delta x_k) \to 0, \quad (y_k, s_k) \to (x_0, t_0), \quad u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) \to \overline{u}(x_0, t_0),
\]

\((y_k, s_k)\) is a global maximum point of \(u^{(\Delta t_k, \Delta x_k)}(\cdot, \cdot) - \phi(\cdot, \cdot)\).

Denoting by \(\xi_k = u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) - \phi(y_k, s_k)\), we have

\[
\xi_k \to 0 \quad \text{and} \quad u^{(\Delta t_k, \Delta x_k)}(x, t) \leq \phi(x, t) + \xi_k,
\]

for all \((x, t) \in \mathbb{R}^n \times [0, T]\). By the definition of \(u^{(\Delta t_k, \Delta x_k)}\), the hypotheses (5.2.2), (5.3.1)

\[
S(\Delta t, \Delta x, n_k, j_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k) \leq 0.
\]

Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:

\[
0 \geq \liminf_{k \to \infty} \frac{S(\Delta t_k, \Delta x_k, n_k, j_k, \phi(y_k, s_k) + \xi_k, \tilde{\phi} + \xi_k)}{\rho_k(\Delta t, \Delta x)} \geq \liminf_{(n \Delta t, j \Delta x) \to (t, x)} \liminf_{\xi \to 0} \frac{S(\Delta t, \Delta x, n, j, \phi(y, s) + \xi, \tilde{\phi} + \xi)}{\rho(\Delta t, \Delta x)} \geq \partial_t \phi + F^*_x(x, t, \phi, D\phi, D^2\phi),
\]

which is the desired result, because of the assumption \(\overline{u}(x_0, t_0) = \phi(x_0, t_0)\).

### 5.3.1 Finite Difference Schemes for the Black & Scholes Equation

Let us consider the Black and Scholes equation (1.1.3). It turns out to be easy to express the option price in term of the logarithmic value of the stock price: we therefore use the logarithmic change of variable \(x = \log(S)\) and the unknown function \(u(x, \tau) = V(e^x, \tau)\). Moreover, to maintain the convention used in Chapter 4, we change the time to maturity \(t = T - \tau\). Then, the function \(u(x, t)\) is the smooth solution of the Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t} - \frac{1}{2\sigma^2} \frac{\partial^2 u}{\partial x^2} - (r - \frac{1}{2} \sigma^2) \frac{\partial u}{\partial x} + ru = 0 \quad \forall (x, t) \in \mathbb{R} \times [0, T), \\
u(x, 0) = \phi(e^x) \quad \forall x \in \mathbb{R},
\end{cases}
\]

(5.3.2)

This equation is a particular case of the convection-diffusion equation introduced in (4.4.1), with the additional linear term \(ru\). The analysis given in Section 4.4,
can then be extended to this case. Using the notations of the latter paragraphs, the explicit 3-point approximation (4.4.4) to the equation (5.3.2) assumes the equivalent form

\[ \tilde{Q}(h,k,m,n,u^m_n+1,u^m_n-1,u^m_n) = \frac{u^m_n+1 - u^m_n}{k} + a \frac{u^m_n+1 - u^m_n-1}{2h} \]

\[ - \left( \frac{q}{2k} + \frac{b}{h^2} \right) (u^m_n+1 - 2u^m_n + u^m_n-1) + cu^m_n = 0, \]

where \( a = -(r - \sigma^2/2) \), \( b = \sigma^2/2 \) and \( c = r \).

This approximation is monotone, then \( l^\infty \)-stable (see Section 4.4 and 4.1.3), under the condition

\[ \frac{|a|k}{h} \leq q + \frac{2bk}{h^2} + ck \leq 1. \]

The most elementary way to avoid the CFL conditions is to use an implicit scheme in time, such as a Crank-Nicholson scheme (see Section 4.3).

In Chapter 7, we shall give an application of high order Implicit-Explicit (IMEX) schemes to this problem, as a particular case of parabolic integro-differential problems.

Here, we shall introduce the problem of numerical boundary conditions. Notice that in general, the numerical boundary condition coupled with a particular scheme can be unstable (see for instance [61, Section 11.3] for further details).

Since the problem (5.3.2) is defined in the all real line, to calculate its numerical solution it is necessary to truncate the problem domain. Moreover, once we have found a given domain, we still need to use some approximations of the solution at the boundary. This is because the scheme requires grid points to the left and right of \((t_n,x_m)\) when computing \(u^m_{n+1}\), and at the boundary either \(x_{m-1}\) or \(x_{m+1}\) is not a grid point. Therefore, at \(x_M\), where \(x_M\) is the last grid point on the right, we must use some means other than the scheme to compute \(u^m_{n+1}\), as for the last grid point on the left \(x_0\).

The common approach is to use some asymptotic representation formula for the solution. If the solution of the differential equation (5.3.2) is the price of a European option with strike price \(K\) and maturity \(T\), it can be proved that for call and put options, we have respectively

\[ u_C(x,t) = e^x - Ke^{-r(T-t)} \quad \text{for} \quad x \to \infty \]

\[ u_P(x,t) = Ke^{-r(T-t)} - e^x \quad \text{for} \quad x \to 0^+. \]

This two approximations can be used to compute the boundary values, \(u^m_M\) and \(u^m_0\) respectively (notice that for \(x \to 0^+\) the price of a call option will be zero as for the put option for \(x \to \infty\)).

However, notice that as to keep stability, the boundary approximation may involve challenging difficulties to hold the high order accuracy of schemes.
Chapter 6

Numerical approximations: the Integro-Differential case

As we have shown in the previous chapter, a great deal has been done for the numerical approximation of viscosity solutions, starting from [24]. For second order problems we have remarked the fundamental paper by Barles and Souganidis [11], who first showed convergence results for a large class of numerical schemes to the solution of fully nonlinear second order elliptic or parabolic PDE. On their track, in this chapter we extend their arguments to the class of numerical schemes for integro-differential problems of the form (0.0.1)-(0.0.2).

Let us also recall that in the framework of linear problems with constant coefficients, the integral term (0.0.2) was already considered in [6]. In that paper the authors proposed to use an operator splitting method compared with the drawbacks of a pure Crank-Nicholson one. In that context, the method is shown to be quite effective: it has a lighter computational burden and allows to couple the differential part, with an implicit finite difference method, and the integral part, with an FFT method. The FFT method requires a constant grid step, but it could diminish the numerical precision of the scheme in some areas; it is possible to overcome this difficulty using an asymptotic profile of the solution or a particular feature of the integral operator. Although a closer discussion of this method is done in Section 6.5.2, we notice here that its rigorous assessing, as well as its extension to fully nonlinear strongly degenerate problems, which are the main objectives of our investigation, seem to be quite difficult and have still to be done.

Another difficult problem stems from the nonlocal nature of the integral term. Actually, it is necessary to truncate the problem domain on one hand, and the integral domain on the other. As $\mu_{x,t}$ is a bounded measure, for a fixed $\nu > 0$ we can choose a bounded computational domain $D_\nu$ for the integral term, such that

$$\left| \int_{\mathbb{R}^N} \mu_{x,t}(dz) - \int_{D_\nu} \mu_{x,t}(dz) \right| < \nu,$$

and we can consider a new problem with $\mathcal{I}_\nu u = \int_{D_\nu} M(u(x + z, t), u(x, t)) \mu_{x,t}(dz)$
instead of \( Iu \); after that, we have to truncate the domain of the problem.

Unfortunately, due the non-local nature of the integral term, once we have found a given domain, still we need to use some approximation of the solution in a larger computational domain. The usual solution consists in using some asymptotic representation formula for the solution.

Here we try a different approach. First we notice that our original problem can be well approximated, by a pure differential problem with an artificial diffusion. We apply this remark to implement an effective numerical boundary condition, which reduces the error due to the truncation.

This chapter is organized as follows: first of all in Section 6.1 we give the main convergence result; Section 6.2 is devoted to the study of the numerical computation of the integral term and the study of the scheme in the one dimensional case; in Section 6.4 we show that we can approximate the integro–differential operator, by a suitable diffusive differential model.

In Section 6.5 we focus on numerical simulations: after a short review of some recent numerical methods for the linear PIDE, we give a quite complete description of an explicit scheme. In particular in Subsection 6.5.2, we deal with the numerical boundary conditions proving that, under proper assumptions on the measure \( \mu_{x,t} \) and on the grid steps, the whole scheme is second order accurate.

We conclude, in Section 6.6, by presenting some numerical examples. First we deal with the classical Morton equation. A two dimensional problem is studied in Subsection 6.6.2, and in Subsection 6.6.3, we consider the large investor nonlinear case.

### 6.1 A general convergence result

Let the assumptions of Section 3.2 be given; we want to approximate the viscosity solution of the following problem:

\[
\begin{aligned}
\partial_t u + F(x, t, u, Iu, Du, D^2u) &= 0, \\
u(x, 0) &= u_0(x),
\end{aligned}
\]  

where the function \( F \in C(\mathbb{R}^N \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N, \mathbb{R}) \), satisfies the hypothesis \textbf{F1-F3} in Section 3.2 and where \( \mathcal{S}_N \) is the set of the symmetric \( N \times N \) matrices.

We define a numerical grid in \( \mathbb{R}^N \times (0, T) \) using the following notation: \( h = (h_1, ..., h_N) \) is the spatial grid size, \( k \) is the time grid size, \( (x_j, t_n) = (jh, nk), j \in \mathbb{Z} \) and \( n \in \mathbb{N} \), are the grid points, \( v^n_j \) is the value of the function \( v \), defined on the grid or defined for continuously varying \( (x, t) \), at the grid point \( (x_j, t_n) \) and \( \tilde{v} \) is the vector of \( v \) values, \( (v^n_j)_{j,n} \) for \( j \) and \( n \) varying on a subset of \( \mathbb{Z} \) and \( \mathbb{N} \) respectively.

A numerical scheme approximating (6.1.1) can be written as

\[
Q(h, k, j, n, v^n_j, Iu, \tilde{u}) = 0,
\]  

(6.1.2)
where \( \mathcal{I}_h \tilde{u} \) denotes the integral approximation. We want to prove that, under suitable conditions, this scheme converges to the solution of the problem (6.1.1), provided that this problem satisfies proper conditions.

**Properties of the scheme**

**H1** Monotonicity of the approximating integral.
If \( \tilde{u} \geq \tilde{v} \) and \( u^n_j = v^n_j \) we have the following inequality:
\[
\mathcal{I}_h \tilde{u} \geq \mathcal{I}_h \tilde{v};
\] (6.1.3)

**H2** Stability.
For all \( h, k \) a solution \( \hat{u} \) does exist that is bounded independently from \( (h, k) \); (6.1.4)

**H3** Consistency.
For all \( \phi \in C^\infty_b(\mathbb{R}^N \times [0, T]) \) and for all \( (x, t) \in \mathbb{R}^N \times (0, T) \) we have:
\[
\lim \inf_{(h, k) \to 0} \limsup_{(j, h, nk) \to (x, t)} \lim_{\xi \to 0} \frac{Q(h, k, j, n, \phi^n_j + \xi, \mathcal{I}_h (\hat{\phi} + \xi), \hat{\phi} + \xi)}{\rho(h, k)} \geq \partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2u);
\] (6.1.5)

\[
\lim \sup_{(h, k) \to 0} \liminf_{(j, h, nk) \to (x, t)} \lim_{\xi \to 0} \frac{Q(h, k, j, n, \phi^n_j + \xi, \mathcal{I}_h (\hat{\phi} + \xi), \hat{\phi} + \xi)}{\rho(h, k)} \leq \partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2u);
\] (6.1.6)

**H4** Monotonicity.
If \( \tilde{u} \geq \tilde{v} \) and \( u^n_j = v^n_j \) for all \( h, k \geq 0 \) and \( 1 \leq n \leq N \), we have:
\[
Q(h, k, j, n, u^n_j, \mathcal{I}_h \tilde{u}, \tilde{u}) \leq Q(h, k, n, j, v^n_j, \mathcal{I}_h \tilde{v}, \tilde{v}).
\] (6.1.7)

**Remark 6.1.1** The theory of numerical approximation of fully nonlinear degenerate parabolic problems, given in Chapter 5 could be considered as a special case of the present one.

**Properties of the equation**

**H5** Maximum Principle or Strong Uniqueness Property
Let \( u_0 \in C(\mathbb{R}^N) \cap L^\infty_{\| \cdot \|}(\mathbb{R}^N) \) be the initial data of (6.1.1), such that there exists \( m > 0 \):
\[
\|u_0(x)\| \leq Be^{\alpha \|x\|} \text{ for } n_0 \leq m;
\]
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if \( u, v \in \bigcup_{n<m} L^\infty(0, T; L^\infty(\mathbb{R}^N)) \) are an \( \text{USC}^T \) subsolution and a \( \text{LSC}^T \) supersolution of (6.1.1) respectively, then

\[
\ u \leq v \text{ on } \mathbb{R}^N \times [0, T].
\]

Under these assumptions we shall prove our main theoretical result.

**Theorem 6.1.2** Let assumption \((H1)-(H5)\) hold true. Then, as \((h, k) \to 0\), the solution \( \bar{u} \) of the scheme (6.1.2) converges locally uniformly to the unique continuous viscosity solution of the problem (6.1.1)

**Proof.** Let \( \underline{u}, \bar{u} \in \bigcup_{n<m} L^\infty(0, T; L^\infty(\mathbb{R}^N)) \) be defined by:

\[
\underline{u}(x, t) = \lim \inf_{(\Delta t, \Delta x) \to 0} (\Delta t, j) u^n_j,
\]

\[
\bar{u}(x, t) = \lim \sup_{(\Delta t, \Delta x) \to 0} (\Delta t, j) u^n_j.
\]

We want to prove that \( \underline{u} \) and \( \bar{u} \) are respectively supersolution and subsolution of the problem (6.1.1). If this claim is proved to be true, then by definition we have \( \underline{u} \leq \bar{u} \), while the other inequality holds because of lower semicontinuity of \( \underline{u} \) and upper semicontinuity of \( \bar{u} \), hence:

\[
\underline{u} = \bar{u} = u,
\]

is the unique continuous solution of the problem (6.1.1). This result, together with the definition of \( \underline{u} \) and of \( \bar{u} \) leads to the local uniform convergence of the solution of the scheme to the solution of the problem. To prove the claim, we will consider only the case of \( \bar{u} \), the other being the same. We want to prove that \( \bar{u} \in \text{USC}^T \) and that it is a subsolution for the problem, i.e. for all \( \phi \in C^{2,1}(\mathbb{R}^N \times [0, T]) \) such that \( \bar{u} - \phi \) has a local maximum in \( (x_0, t_0) \) we have:

\[
\partial_t \phi(x_0, t_0) + F(x_0, t_0, \bar{u}(x_0, t_0), \nabla \phi(x_0, t_0), D\phi(x_0, t_0), D^2 \phi(x_0, t_0)) \leq 0.
\]

We start by proving that \( \bar{u} \in \text{USC}^T \).

1. \( \bar{u} \) is upper semicontinuous: it follows from the definition itself: we want to prove that the following inequality holds:

\[
\lim \sup_{(y, s) \to (x, t)} \bar{u}(y, s) \leq \bar{u}(x, t),
\]
By definition:
\[ p(y, s) = \limsup_{(\Delta t, \Delta x) \to 0} u^n_j; \]
therefore, by definition of limsup, there exist an \( \varepsilon > 0 \) and \( (n, j) \) such that:
\[ p(y, s) - \varepsilon \leq p_j^n; \]
now, taking limsup for \( (\Delta t, \Delta x) \to 0 \) and \( (n \Delta t, j \Delta x) \to (t, x) \), we obtain:
\[ p(y, s) - \varepsilon \leq p(x, t); \]
Now, as \( \varepsilon \) is arbitrarily chosen, we obtain the desired result.

2. \( p \) is locally bounded: by definition it is obtained by the solutions of the scheme. By hypothesis (6.1.4) they are bounded, independently from \( \Delta t, \Delta x \). So let \( K \in \mathbb{R}^N \times [0, T] \) be a compact set, then there exists a constant \( A_K \) such that:
\[ |p^n_j| \leq A_K \quad \forall n, j \quad \text{s.t.} \quad (n \Delta t, j \Delta x) \in K \Rightarrow |p(x, t)| \leq A_K \quad \forall (x, t) \in K; \]

3. \( M(p(x + z, t), p(x, t)) \) has an upper \( \mu \)-bound in \( (x, t) \). From the hypotheses on \( M \), it clearly follows that \( M \) is a Lipschitz function with constant \( c \), so we have:
\[ M(p(x + z, t), p(x, t)) \leq c|p(x + z, t) - p(x, t)|; \]
it easy to show that in a compact neighborhood \( V_{x,t} \) of \( (x, t) \) we have:
\[ M(p(x + z, t), p(x, t)) \leq 2cA_{V_{x,t}}. \]
It is then sufficient to choose \( \Phi \) as a constant to be the wanted \( \mu \)-bound.

We have proved that \( p \in USC^T \), and now we need only to prove that \( p \) is a viscosity subsolution. To this aim let \( (x_0, t_0) \) be a global strict maximum for \( p - \phi \) on \( \mathbb{R}^N \times [0, T] \) for some \( \phi \in C_b^\infty(\mathbb{R}^N \times [0, T]) \). We could assume that \( p(x_0, t_0) = \phi(x_0, t_0) \) and that:
\[ p(x, t) - \phi(x, t) \leq 0 = p(x_0, t_0) - \phi(x_0, t_0) \quad \text{in} \quad \mathbb{R}^N \times [0, T]. \]
From these hypothesis follows that there exists a sequence \( (\Delta t_k, \Delta x_k) \in \mathbb{R}^+ \) and \( (y_k, s_k) \in \mathbb{R}^N \times [0, T] \) such that, as \( k \to \infty \):
\[ (\Delta t_k, \Delta x_k) \to 0, \quad (y_k, s_k) \to (x_0, t_0), \quad u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) \to p(x_0, t_0), \]
\[ (y_k, s_k) \text{ is a global maximum point of} \quad u^{(\Delta t_k, \Delta x_k)}(\cdot, \cdot) - \phi(\cdot, \cdot). \]
Denoting by \( \xi_k = u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) - \phi(y_k, s_k) \), we have
\[ \xi_k \to 0 \]
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\[ u(\Delta t_k, \Delta x_k)(x, t) \leq \phi(x, t) + \xi_k, \]

for all \((x, t) \in \mathbb{R}^N \times [0, T]\). By the definition of \(u(\Delta t_k, \Delta x_k)\), the hypotheses (6.1.7), (6.1.8) and the hypotheses on \(M\) we obtain:

\[ Q(\Delta t_k, \Delta x_k, n_k, j_k, \phi(y_k, s_k) + \xi_k, \mathcal{I}_{n_k,j_k} (\tilde{\phi} + \xi_k), \tilde{\phi} + \xi_k) \leq 0. \]

Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:

\[ 0 \geq \liminf_{k \to \infty} Q(\Delta t_k, \Delta x_k, n_k, j_k, \phi(y_k, s_k) + \xi_k, \mathcal{I}_{n_k,j_k} (\tilde{\phi} + \xi_k), \tilde{\phi} + \xi_k) \rho(\Delta t, \Delta x) \geq \liminf_{(\Delta t, \Delta x) \to 0} \frac{Q(\Delta t, \Delta x, n, j, \phi(y, s) + \xi, \mathcal{I}(\tilde{\phi} + \xi), \tilde{\phi} + \xi)}{\rho(\Delta t, \Delta x)} \]

\[ \geq \partial_t \phi + F(x, t, \phi, \mathcal{I}\phi, D\phi, D^2\phi), \]

which is the desired result, because of the assumption \(\mathcal{I}(x_0, t_0) = \phi(x_0, t_0)\).

6.2 The numerical approximation of the integral term

According to the classical theory of approximated integration, see for instance [27], we use the compound Newton-Cotes formulas to approximate the integral term on the interval \([a, b]\):

\[ (R_S)(f) = \frac{b - a}{2S} \sum_{s=0}^{S-1} \sum_{i=1}^{\rho} \alpha_i f(x_{is}) \approx \int_a^b f(x)dx, \quad (6.2.1) \]

where \(S\) is the number of subinterval in which we have divided \([a, b]\), \(a = y_0 < y_1 < \ldots < y_S = b\), \(\rho\) is the number of point in each subinterval \([y_s, y_{s+1}]\) and

\[ x_{is} = y_s + \frac{b - a}{2S}(1 + t_i), \quad s = 0, \ldots, S - 1. \]

The errors occurred in approximate integration formulas are conventionally expressed in terms of the higher derivatives of the integrand function \(f\) and are valid only if the integrand is sufficiently smooth. It is a feature of the Newton-Cotes formulas that, if the number of point is \(2k - 1\) or \(2k\), the error is of the form \(E_R(f) = ch^{2k+1} f^{2k}(\xi)\), for \(a < \xi < b\). To obtain the error estimates when the integrand function \(f\) has a low-order continuity, we can approximate the function \(f\) by a suitable polynomial, according to the following standard result, see for instance [27].
Proposition 6.2.1 Let \( f(x) \) be of class \( C[a, b] \), and let \( \omega(\delta) \) be its modulus of continuity. Then for each \( n \in \mathbb{N} \), there exists a polynomial of degree \( \leq n \), \( p_n(x) \), such that
\[
| f(x) - p_n(x) | \leq 2\omega\left( \frac{b-a}{2^n} \right), \quad a \leq x \leq b.
\]
If \( f(x) \) is of class \( C[a, b] \) and has a bounded derivative,
\[
| f'(x) | \leq M, \quad a \leq x \leq b,
\]
then for each \( n \in \mathbb{N} \) there exists a polynomial \( p_n(x) \) of degree \( \leq n \) such that
\[
| f(x) - p_n(x) | \leq \frac{3(b-a)M}{n}, \quad a \leq x \leq b.
\]

6.3 The numerical approximation in the one dimensional case

Here we want to consider the integro-differential equation (3.2.1) in one dimension. We suppose \( F \) to be linear in the integral part,
\[
\partial_t u + F(x, t, u, Du, D^2u) - I u = 0. \tag{6.3.1}
\]
The first step to approximate the integral operator (0.0.2) using the numerical integrations formula described in the previous section is to truncate the integral domain; let us choose the interval \([z_m, z_M]\) such that
\[
\int_{z_m}^{z_M} \mu_{x,t}(dz) \approx \int_{-\infty}^{+\infty} \mu_{x,t}(dz) - \nu = 1 - \nu, \quad \nu \ll 1. \tag{6.3.2}
\]
Assume \( u(\cdot, t) \in L^\infty(\mathbb{R}) \) and let \( U \) be its \( L^\infty \)-norm. The error due to the truncation of the domain is estimated as follows.
\[
\left| \int_{-\infty}^{+\infty} M(u(x+z, t), u(x,t))\mu_{x,t}(dz) - \int_{z_m}^{z_M} M(u(x+z, t), u(x,t))\mu_{x,t}(dz) \right|
\leq c \int_{-\infty}^{z_m} | u(x+z, t) - u(x,t) | \mu_{x,t}(dz) + c \int_{z_M}^{+\infty} | u(x+z, t) - u(x,t) | \mu_{x,t}(dz)
\leq 2Uc \left[ \int_{-\infty}^{z_m} \mu_{x,t}(dz) + \int_{z_M}^{+\infty} \mu_{x,t}(dz) \right] \tag{6.3.3}
\]
\[
= 2Uc \int_{\mathbb{R} - [z_m, z_M]} \mu_{x,t}(dz) = 2Uc\nu.
\]
6.3.1 The case of the Gaussian distribution

In the estimate (6.3.3), we have supposed that the function $u(\cdot, t)$ is $L^\infty(\mathbb{R})$, but it is possible to suppose even more regularity, under particular hypothesis on the Radon measure. Let consider, for example, on the track of Merton (see Example 1.2.1), the Gaussian distribution. It leads to a probability measure with the property of being symmetric with respect to the origin of the real line. In this case it is possible to assume that $u(\cdot, t) \in \text{Lip}(\mathbb{R})$ with constant $L$; we remember that in that particular case, the integral term is:

$$Iu = \lambda \int_{-\infty}^{+\infty} [u(x + z, t) - u(x, t)]\mu_{x,t}(z)dz,$$

where $\mu_{x,t} = \Gamma_\delta$ is the Gaussian probability density:

$$\mu_{x,t}(dz) := \Gamma_\delta(z)dz = \frac{1}{\sqrt{2\pi} \delta}e^{-\frac{z^2}{2\delta^2}}dz.$$

As it was previously shown, the calculation of the integral term could be simplified by considering a finite interval instead of the whole real line. Thanks to the particular shape of the density measure $\Gamma_\delta$, we can select the finite interval considering only such points for which the density has a significant value and this choice would not introduce big errors. Choose a parameter $\varepsilon > 0$ and select the interval $[z_m, z_M]$ as the set of all the points $z$ that verify:

$$\Gamma_\delta(z) \geq \varepsilon \iff \frac{1}{\sqrt{2\pi} \delta}e^{-\frac{z^2}{2\delta^2}} \geq \varepsilon;$$

by simple calculation we can derive $z_m$ and $z_M$:

$$-\sqrt{-2\delta^2 \log(\varepsilon \delta \sqrt{2\pi})} \leq z \leq \sqrt{-2\delta^2 \log(\varepsilon \delta \sqrt{2\pi})}.$$

As $\Gamma_\delta$ is a symmetric function with respect to its axis (that in this case is the line $z = 0$), we define:

$$z_M = \sqrt{-2\delta^2 \log(\varepsilon \delta \sqrt{2\pi})}, \quad z_m = -z_M.$$

Under these hypotheses we have the following estimate:

$$\left| \int_{-\infty}^{+\infty} M(u(x + z, t), u(x, t))\Gamma_\delta(dz) - \int_{z_m}^{z_M} M(u(x + z, t), u(z, t))\Gamma_\delta(dz) \right|$$

$$\leq L \left( \int_{-\infty}^{z_m} |z|\Gamma_\delta(dz) + \int_{z_M}^{+\infty} |z|\Gamma_\delta(dz) \right),$$
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therefore

\[ \int_{-\infty}^{z_m} |z| \Gamma_\delta (dz) + \int_{z_M}^{+\infty} |z| \Gamma_\delta (dz) \]
\[ = 2 \int_{z_M}^{+\infty} z \frac{1}{\sqrt{2\pi}\delta} \exp \left( -\frac{z^2}{2\delta^2} \right) dz = \frac{2\delta^2}{\sqrt{2\pi}} \exp \left( -\frac{z^2}{2\delta^2} \right) = 2\delta^2 \varepsilon. \quad (6.3.4) \]

Let us now apply the compound rule (6.2.1) to the truncated integral.

\[ I_h u = \lambda (R_S) (M \Gamma_\delta) = \lambda \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^{p} \alpha_i M(u(x + z_{is}, t), u(x, t)) \Gamma_\delta(z_{is}). \quad (6.3.5) \]

Since the function \( g(z) = M(u(x + z, t), u(x, t)) \Gamma_\delta(z) \) has a low-order continuity, to get an error estimate for the approximation (6.3.5), we apply Proposition 6.2.1.

Recall that for a generic function \( f \) we have

\[ \mathcal{E}_R(f) = \int_a^b f(x)dx - R(f) = \int_a^b (f(x) - p_n(x))dx + \int_a^b p_n(x)dx - R(f) \]
\[ = \int_a^b (f(x) - p_n(x))dx + R(p_n - f). \]

Then

\[ | \mathcal{E}_R(f) | \leq \left( (b - a) + \sum_{i=0}^{p} |\alpha_i| \right) | f(x) - p_n(x) |. \]

If \( \alpha_i > 0 \), using formula (6.2.1), we obtain

\[ | \mathcal{E}_R(f) | \leq 2(b - a) | f(x) - p_n(x) |. \]

An \((R_S)\) compound rule, applied to our function \( g \), yields

\[ \mathcal{E}_{R_S}(g) = \sum_{s=0}^{S-1} \mathcal{E}_R(g) = \sum_{s=0}^{S-1} 2 \left( \frac{b - a}{S} \right) | g(x_s) - p_n(x_s) |. \]

Then there exists a polynomial of degree \( \leq Sp, p_{Sp}(z) \), such that

\[ | g(z) - p_{Sp}(z) | \leq 2\omega \left( \frac{z_M - z_m}{2Sp} \right). \]

There follows

\[ \left| \int_{z_m}^{z_M} g(z)dz - (R_S)(g(z)) \right| \leq \int_{z_m}^{z_M} | g(z) - p_{Sp}(z) | dz + \left| (R_S)(p_{Sp}(z) - g(z)) \right| \quad (6.3.6) \]
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\[
\leq \left( (z_M - z_m) + \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^{p} |\alpha_i| \right) 2\omega \left( \frac{z_M - z_m}{2Sp} \right)
\]

\[
\leq 2(z_M - z_m) \left( 1 + \frac{1}{2} \sum_{i=0}^{p} |\alpha_i| \right) \omega \left( \frac{h}{2} \right),
\]

where \( \omega \) is the modulus of continuity for \( g \).

6.3.2 Check of the hypotheses of Theorem 6.1.2 for the integral part

We have first to approximate the differential operator \( \partial_t + F \): we take a numerical scheme \( \tilde{Q} \) that verifies the convergence (differential) conditions (S1)-(S4) of Section 5.2 of [11]. In particular, to keep the order of the convergence of the integration formula (6.3.5), we assume that the space discretization grid of the numerical operator \( \tilde{Q} \) coincides with the integral one, i.e. we set the common space step \( h \) such that

\[
h \leq \frac{z_M - z_m}{p \cdot S}.
\]

Then the approximation of the integro-differential equation (6.3.1) is given by:

\[
Q(h,k,j,n,u^n_j,\mathcal{I}_h\tilde{u},\tilde{u}) = \tilde{Q}(h,k,j,n,u^n_j,\tilde{u}) - \mathcal{I}_h\tilde{u} = 0,
\]

We want to show that under the above assumption, this scheme satisfies conditions (6.1.3)-(6.1.7).

1. Monotonicity of the approximating integral

Since the function \( M \) is such that

\[
M(u, w) \leq M(v, w), \text{ if } u \leq v,
\]

to get the monotonicity of the integral approximation it is sufficient that the weights \( \alpha_i \) are greater than zero for all \( i \). Clearly, if \( \tilde{u} \leq \tilde{v} \) and \( u^n_j = v^n_j \), we have

\[
\lambda \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^{p} \alpha_i M(u^n_j(x_j + z_{is}), u^n_j) \Gamma_\delta(z_{is})
\]

\[
\leq \lambda \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^{p} \alpha_i M(v^n_j(x_j + z_{is}), v^n_j) \Gamma_\delta(z_{is}),
\]

for all \( j \in \mathbb{Z} \) and \( n \in \mathbb{N} \).

2. Stability

It is a trivial consequence of the \( \tilde{Q} \) stability (S2) and the monotonicity of the integral approximation.
3. Consistency

Let \( \phi \in C^\infty(\mathbb{R} \times (0, T)) \), from the consistency condition (S3) on \( \tilde{Q} \), we get the following inequality:

\[
\lim_{(h,k) \to 0} \frac{\tilde{Q}(h, k, j, n, \phi^n + \xi, \phi + \xi) - \mathcal{I}_h(\tilde{\phi} + \xi)}{\rho(h, k)} \geq \partial_t u + F(x, t, u, \mathcal{D}u, \mathcal{D}^2u) - \lim_{(h,k) \to 0} \frac{\mathcal{I}_h(\tilde{\phi} + \xi)}{\rho(h, k)}.
\]

From the error estimate of the integral approximation (6.3.6), we have

\[
\lim_{(h,k) \to 0} \frac{\mathcal{I}_h(\tilde{\phi} + \xi)}{\rho(h, k)} = \mathcal{I}_u - \lim_{(h,k) \to 0} \frac{\mathcal{E}_{\mathcal{R}_S}(\tilde{\phi} + \xi)}{\rho(h, k)} = \mathcal{I}_u.
\]

then, we get condition (6.1.5). Condition (6.1.6) follows by analogous considerations.

4. Monotonicity

It is a trivial consequence of the \( \tilde{Q} \) monotonicity (S1) and the monotonicity of the integral approximation (point 1).

Remark 6.3.1 In our numerical test, we have always considered a Radon measure absolutely continuous with respect to the Lebesgue measure, i.e:

\[
\mu_{x,t}(dz) = \lambda \Gamma_\delta(z) dz.
\]

It is even possible to consider a discrete measure, for example the Dirac measure:

\[
\mu_{x,t}(dz) = \delta_{z_0}(z) dz.
\]

In that case the numerical approximation is even simpler, thanks to the absence of the integral term.

6.4 The diffusive effect of the integral operator

An important point in the numerical simulation for the problem we have presented, is the behaviour of the solution at the limiting point of the truncated numerical
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In this particular framework, the presence of the integral term which convolute “internal” and “external” points requires a particular tool to deal with such a difficulty. One possibility is to look at the particular form of the integral term \( \mathcal{I}u \) with respect to the Gaussian parameter \( \delta \): we show that a convenient way to deal with the integral operator is to replace it (locally) by an effective diffusion term. This result will be useful in the numerical simulations, as is shown next in Subsection 6.5.2. The following discussion, which is presented only in the linear case, has the main purpose of rigorously investigating the error generated by this approximation.

Let us consider the two following one dimensional equations, for \((x,t) \in \mathbb{R} \times (0,T)\):

\[
\begin{align*}
    u_t + au_x - bu_{xx} + cu &= \mathcal{I}u, \quad (6.4.1) \\
    v_t + av_x - bv_{xx} + cv &= \frac{\lambda \delta^2}{2} v_{xx}, \quad (6.4.2)
\end{align*}
\]

with the same initial condition

\[ u(x,0) = v(x,0) = u_0(x), \quad x \in \mathbb{R}. \]

It is possible to prove that, under proper hypotheses on the density distribution \( \Gamma_\delta \) and on the solutions \( u \) and \( v \), the integral problem (6.4.1) is well approximated by the advection-diffusion one (6.4.2).

**Proposition 6.4.1** Let \( u \) be the solution of problem (6.4.1) and \( v \) the solution of problem (6.4.2) with the same initial condition \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Then, if \( \delta \ll 1 \), there holds

\[ ||u - v||_{L^\infty(0,T;L^1(\mathbb{R}))} \leq O(T \delta^3). \]

**Proof.** The function \( w = u - v \) is a solution of the following problem (written in the weak formulation):

\[
\begin{align*}
   &- \int_0^T \int_{-\infty}^{+\infty} \left[ \phi_t(x,t) + a\phi_x(x,t) + b\phi_{xx}(x,t) - c\phi(x,t) \right] w(x,t) dx dt \\
   &= \lambda \int_0^T \int_{-\infty}^{+\infty} \frac{\delta^2}{2} \phi_{xx}(x,t) w(x,t) dx dt \\
   &+ \lambda \int_0^T \int_{-\infty}^{+\infty} u(x,t) \int_{-\infty}^{+\infty} \left[ \phi(x+z,t) - \phi(x,t) - \frac{\delta^2}{2} \phi_{xx}(x,t) \right] \Gamma(z) dz dx dt,
\end{align*}
\]

for every test function \( \phi \in C_0^\infty(\mathbb{R} \times [0,T]) \).

To estimate the inner integral in the second member of the right hand side we can take the Taylor expansion of \( \phi \), which leads to

\[ \phi(x+z,t) - \phi(x,t) - \frac{\delta^2}{2} \phi_{xx}(x,t) \]
we can estimate in terms of the norm of $u$, the error made by using this expansion:

$$
\left| \lambda \int_{0}^{T} \int_{-\infty}^{+\infty} u(x,t) \int_{-\infty}^{+\infty} \left[ \phi(x + z, t) - \phi(x, t) - \frac{z^2}{2} \phi_{xx}(x, t) \right] \Gamma_\delta(z) \, dz \, dx \, dt \right|
\leq \lambda \int_{0}^{T} \int_{-\infty}^{+\infty} u(x,t) \int_{-\infty}^{+\infty} \frac{z^3}{6} \left[ \phi(x + (1 - k)z, t) \right] \Gamma_\delta(z) \, dz \, dx \, dt
\leq \int_{0}^{T} \| \phi(\cdot, t) \|_{C^3([0,T])} \| u(\cdot, t) \|_{L^1(\mathbb{R})} dt \int_{-\infty}^{+\infty} \left| \frac{z^3}{6} \Gamma_\delta(z) \right| \, dz
\leq \frac{\lambda \delta^3}{\sqrt{2\pi}} \| \phi \|_{C^3([0,T])} \| u \|_{L^\infty(0,T;L^1(\mathbb{R}))} = O(T \delta^3).
$$

The right hand side of (6.4.3) can be rewritten taking into account the last estimate:

$$
- \int_{0}^{T} \int_{-\infty}^{+\infty} \left[ \phi_t(x,t) + a \phi_x(x,t) + b \phi_{xx}(x,t) - c \phi(x,t) \right] w(x,t) \, dx \, dt
= \lambda \int_{0}^{T} \int_{-\infty}^{+\infty} \frac{\delta^2}{2} \phi_{xx} w(x,t) \, dx \, dt
+ \lambda \int_{0}^{T} \int_{-\infty}^{+\infty} w(x,t) \int_{-\infty}^{+\infty} \left[ \phi_x(x,t) z + \frac{z^2 - \delta^2}{2} \phi_{xx}(x,t) \right] \Gamma_\delta(z) \, dz \, dx \, dt
+ O(T \delta^3).
$$

Since for the inner integral, there holds

$$
\int_{-\infty}^{+\infty} \left[ \phi_x(x,t) z + \frac{z^2 - \delta^2}{2} \phi_{xx}(x,t) \right] \Gamma_\delta(z) \, dz = 0,
$$

$w$ is just the weak solution to problem

$$
w_t + aw_x - \left( b + \lambda \frac{\delta^2}{2} \right) w_{xx} + cw = O(T \delta^3),
$$

with initial datum

$$
w(x,0) = 0, \quad x \in \mathbb{R},
$$

which yields

$$
\| w \|_{L^\infty(0,T;L^1_{loc}(\mathbb{R}))} \leq O(T \delta^3).
$$

Now, taking a suitable sequence of test functions such that $\text{supp} \ \phi(x) = [-R, R]$, and letting $R \to +\infty$, gives the result. 

6.5 Finite difference methods for the one dimensional jump-diffusion model

In this section we introduce an explicit approximation for the linear PIDE arising from the jump-diffusion models (1.2.12) and we give a convenient way to deal with the problem of the numerical boundary conditions.

First of all it is important to recall that for the pure diffusion Black-Scholes problem, a huge literature exists for the subject of numerical approximation for the linear convection-diffusion equations, see Section 4.4 and Section 5.3.1. We extend the discussion to the PIDE.

we recall that after appropriate logarithmic transformations the Merton problem (1.2.12) becomes

\[
\begin{align*}
 u_t + au_x &= bu_{xx} - cu + \lambda \left( \int_{-\infty}^{\infty} u(x + z, t) \Gamma_\delta(z) dz - u \right), \\
 u(x, 0) &= \psi(x),
\end{align*}
\]

where

\[
a = - (r - \lambda \bar{k} - \frac{1}{2} \sigma^2), \quad b = \frac{1}{2} \sigma^2, \quad c = r, \quad \bar{k} = E(\eta - 1),
\]

and the initial data \( \psi(x) \) is the payoff function of the European contingent claim.

Let be given the exercise price \( E \), we have respectively for the call and the put option,

\[
\psi(x) = (e^x - E)_+ \quad \text{and} \quad \psi(x) = (E - e^x)_+.
\]

As done in (4.3.4), we can write the time approximation of the PIDE (6.5.1) in the following "\( \theta \)-form":

\[
\frac{u_{j}^{n+1} - u_{j}^{n}}{k} + \mathcal{L} \left[ \theta_1 u_j^n + (1 - \theta_1) u_j^{n+1} \right] + \theta_2 \mathcal{I} u_j^n + (1 - \theta_2) \mathcal{I} u_j^{n+1} = 0,
\]

where \( \theta_1, \theta_2 \in [0, 1] \). The choice \( \theta_1 = \theta_2 = 0 \) gives the explicit scheme, while \( \theta_1 = \theta_2 = 1 \) gives an implicit time differencing scheme, unconditionally stable, but not practically feasible. Actually the convolution integral introduces a significant complication for the numerical solution, since it couples grid points over an extended range, leading to a dense system of equations which is hard to be solved. In fact, after discretizing the \( x \)-space into \( N \) points, inversion of a full \( N \times N \) matrices is required; For \( \theta_1 = 1/2, \theta_2 = 0 \) it gives an asymmetric treatment (implicit-explicit) of the differential and integral part. This is a way to avoid dense system, but it is only first order in time.

In the book [26], Tavella and Randall propose an iterative approach to avoid dense system and to increase the convergence order in time. They write the time-discretized equation as

\[
\frac{u^{m+1} - u^n}{k} + \mathcal{L} \frac{u^{m+1} + u^n}{2} +
\]
\[-\lambda \left( \int_{-\infty}^{\infty} \frac{u^m(x+z) + u^n(x+z)}{2} \Gamma_\delta(z) dz - \frac{u^m + u^n}{2} \right) = 0.\]

At each time step, the iteration begins with \(u^m = u^n\), then proceeds by solving for \(u^{m+1}\) and substituting the new \(u^{m+1}\) for \(u^m\). Iteration proceeds until a convergence criterion is met. Here they set \(u^{m+1} \approx u^{n+1}\) and a new time step is begun. Due to the iteration procedure, this method turns out to be computationally heavy and it is still not clear how to select a good stop criterion.

In the article [6], Andersen and Andreasen proposed an FFT-ADI (Fast Fourier Transform - Alternating Directions Implicit) to avoid the conditional stability of explicit methods. The FFT techniques is applied to the convolution integral and coupled with an ADI method where each time step is split into two half steps: the idea is to choose in the time approximation (6.5.2), \(\theta_1 = 1\) and \(\theta_2 = 0\) for the half time step \(t_n \to t_{n+1/2}\) and \(\theta_1 = 0\) and \(\theta_2 = 1\) for \(t_{n+1/2} \to t_{n+1}\). Then, the discrete version of (6.5.1) is

\[
\begin{cases}
(\frac{2}{h} + \mathcal{L})u^{n+\frac{3}{2}} = (\frac{2}{h} - \lambda \Gamma*)u^n \\
(\frac{2}{h} - \lambda + \lambda \Gamma*)u^{n+1} = (\frac{2}{h} - \mathcal{L})u^{n+\frac{1}{2}},
\end{cases}
\]

where \(\Gamma* u^n\) is the FFT approximation of the convolution term. As shown in [6], this scheme has the following good properties: (i) it is unconditionally stable in the von Neumann sense; (ii) for the case of deterministic parameters, the numerical solution of the scheme is locally accurate of order \(O(k^2 + h^2)\); (iii) if \(M\) is the number of time steps and \(N\) is the number of steps in spatial direction, the computational burden is \(O(MN \log_2 N)\).

Notice that this method is just proposed for the linear constant coefficient one dimensional case, namely for the original Merton equation. We point out that the main difference with the scheme (6.5.4) that we will present in next section, is not the FFT approximation of the convolution term. Actually, our integral approximation formula in (6.5.4), can be easily substituted by the FFT technique without changing the general behaviour of the scheme.

Instead, the main feature of that scheme is in an original decomposition to solve the implicit part. Actually, in the second half time step of (6.5.3), the values \(\{u^{n+1}_j\}\) are first computed in the Fourier space as

\[
<u^{n+1}>_j = \frac{<\frac{2}{h} - \mathcal{L}> u^{n+\frac{1}{2}}>_j}{<\frac{2}{h} - \lambda + \lambda \Gamma >}_j,
\]

and then transformed back by the inverse FFT. However, this procedure turns out to be of difficult implementation and even the monotonicity property of the problem is far to be clear. Moreover, due to the nonlinearities and degeneracies of the considered equations, the effectiveness of these methods in the general case has still to be established.
6.5.1 An explicit finite difference method

In this section, we give an exhaustive description of the explicit scheme. To solve the integro-differential equation (6.5.1), first, we truncate the integral domain. As we have previously described in Subsection 6.3.1, we choose the interval \([z_m, z_M]\) such that (6.3.2) holds and we point out that it exists a positive constant \(C\) such that

\[
\int_{-\infty}^{\infty} [u(x + z, t) - u(x, t)] \Gamma_\delta(z) dz = \int_{z_m}^{z_M} u(x + z, t) \Gamma_\delta(z) dz - u(x, t) + C\delta^2 \epsilon.
\]

We apply a compound rule to the integral term and a standard explicit finite-difference scheme for the differential part as done in (5.3.3) for the Black and Scholes equation. Then, our approximation of the equation (6.5.1) is given by,

\[
Q(h, k, j, n, u_{n,j}, I_h \tilde{u}, \tilde{u}) = u_{n,j}^{n+1} - u_{n,j}^n + a\frac{u_{n,j+1}^n - u_{n,j-1}^n}{2h} - \left(\frac{q}{2k} + \frac{b}{h^2}\right) \left( u_{n,j+1}^n - 2u_{n,j}^n + u_{n,j-1}^n \right) + cu_{n,j}^n + \lambda u_{n,j}^n - \lambda \sum_{p \in P} \alpha_p u_{n,j+p}^n (\Gamma_\delta)_p,
\]

where \(P\) is the index set of the integral approximation.

**Proposition 6.5.1** The scheme (6.5.4) is accurate to order \(O\left(h^2 + \frac{qh^2}{2k}\right)\) under the CFL stability condition

\[
\frac{|a| k}{h} \leq \frac{2bk}{h^2} + q \leq 1 - (c + \lambda)k.
\]

**Proof.** The condition (6.5.5) is easily checked by looking at the monotonicity of the function \(Q\). To study the accuracy of the scheme, we use the symbol analysis, see Section 4.2.3. Let \(p(s, \xi)\) be the symbol of the integro-differential operator (6.5.1)

\[
p(s, \xi) = s + ia\xi + b\xi^2 + c - \lambda \left( \int_{z_m}^{z_M} e^{i\xi z} \Gamma_\delta(z) dz + 1 + C\delta^2 \epsilon \right).
\]

Substituting \(u_j^n = e^{skn}e^{ijh\xi}\) in (6.5.4), we get the symbol \(p_{k,h}(s, \xi)\) of the difference scheme,

\[
p_{k,h}(s, \xi) = \frac{e^{sk} - 1}{k} + ia \sin \frac{h\xi}{h} + 2\left( \frac{qh^2}{2k} - b \right) \cos \frac{h\xi}{h^2} - \frac{1}{h^2} + c - \lambda \left( \sum_{p \in P} \alpha_p e^{iph\xi} (\Gamma_\delta)_p - 1 \right).
\]

Taking into account that our integral approximation verifies

\[
\sum_{p \in P} \alpha_p e^{iph\xi} (\Gamma_\delta)_p = \int_{z_m}^{z_M} e^{i\xi z} \Gamma_\delta(z) dz + O(h^l) \quad \text{with} \quad l \geq 2,
\]

we can conclude that the scheme is accurate to order \(O\left(h^2 + \frac{qh^2}{2k}\right)\) under the CFL stability condition.
we have, by the Taylor expansion
\[ p_{k,h}(s, \xi) = s + ia\xi + bk\xi^2 + r - \lambda \left( \int_{z_m}^{z_M} e^{i\xi z} \Gamma_\delta(z) dz - 1 \right) + \frac{qh^2}{2k} \xi^2 + O(k + h^2). \]

Then, we look for a symbol \( r_{k,h}(s, \xi) \) such that the difference \( p_{k,h}(s, \xi) - r_{k,h}(s, \xi)p(s, \xi) \) gives the order of accuracy. We have that \( r_{k,h}(s, \xi) = 1 + o(1) \) and
\[ p_{k,h}(s, \xi) - p(s, \xi) = + \frac{qh^2}{2k} \xi^2 + \lambda C\delta^2 \epsilon + O(k + h^2). \]

Since \( \epsilon \ll 1 \) and, from the CFL condition (6.5.5), \( k = O(h^2) \), the scheme is accurate of order \( O\left(h^2 + \frac{qh^2}{2k}\right) \).

### 6.5.2 Numerical boundary conditions

To apply the scheme (6.5.4) we have to specify a numerical bounded domain. Let \( \Omega \subset \mathbb{R} \) be the interval where we want to calculate the numerical solution. We set

\[ \Omega_h = \{ x_j = jh, j \in \mathbb{Z} \mid x_j \in \Omega \}, \]

and we define the numerical domain \( \bar{\Omega} \) for the problem (6.5.1) in the following way. For every fixed \( x \), set

\[ \Omega_x = \{ z \in [z_m, z_M] \mid x + z \in \Omega \} \]

and

\[ \bar{\Omega} = \Omega \cup \{ \cup_{x \in \Omega} \Omega_x^C \}. \]

We stress on that, since the integral is a nonlocal term, its approximation will be split in two parts. For every \( x \in \Omega \) fixed, we integrate on the union of the "inside" set \( \Omega_x \) and the "outside" set \( \Omega_x^C \). Since accurate representation of the integral term will generally require a very wide grid, the "outside" set \( \Omega_x^C \) must contains many grid points. Then, as we have defined our numerical problem on \( \Omega \), we need a limiting form for the solution \( u \) on the external set \( \Omega_x^C \). If \( v(x, t) \) is any given analytic approximation of \( u(x, t) \), the integral term will be approximated by

\[ \mathcal{I}u(x) = \lambda \left( \int_{-\infty}^{\infty} u(x + z, t) \Gamma_\delta(z) dz - u(x, t) \right) \]

\[ \approx \lambda \left( \int_{\Omega_x} u(x + z, t) \Gamma_\delta(z) dz + \int_{\Omega_x^C} v(x + z, t) \Gamma_\delta(z) dz - u(x, t) \right). \]

If the option price is linear in \( e^x \), the simplest choice is to use as approximation function \( v \) the payoff function \( \psi \).
To give a more general scheme, not depending on the initial data of the problem or on a special form of the solution, we approximate on the external set \( \Omega^c \) the problem (6.5.1) by the diffusive one (6.4.2). We define

\[
  j_- = \inf \{ jh \in \Omega_h \}, \quad j_+ = \sup \{ jh \in \Omega_h \},
\]

\[
P_{in} = P \cap \{ j_-^-, ..., j_+^+ \}, \quad P_{out} = P - P_{in}.
\]

We modify the scheme (6.5.4) with \( q = 0 \) fixed, as follows

\[
v_{j}^{n+1} = kw_{-1}v_{j-1}^n + \left( 1 - kw_0 \right)v_j^n + kw_1v_{j+1}^n
\]

\[
+ \lambda k \left[ h \sum_{p \in P_{in}} \alpha_p v_{j+p}^n(\Gamma_\delta)_p + h \sum_{p \in P_{out}} \alpha_p \tilde{v}_{j+p}^n(\Gamma_\delta)_p - v_j^n \right],
\]

(6.5.6)

where

\[
w_{-1} = \frac{b}{h^2} + \frac{a}{2h}, \quad w_0 = \frac{2b}{h^2} + c, \quad w_1 = \frac{b}{h^2} - \frac{a}{2h},
\]

(6.5.7)

and where the values \( \{ \tilde{v}_i^n \} \) are given by the approximation of the diffusive equation (6.4.2) with a general diffusion coefficient \( D \), to be fixed later,

\[
\tilde{v}_i^n = kw_{-1}v_{i-1}^{n-1} + \left( 1 - kw_0 \right)v_i^{n-1} + kw_1v_{i+1}^{n-1}
\]

\[
+ \lambda k \frac{D}{h^2} \left( v_{i-1}^{n-1} - 2v_i^{n-1} + v_{i+1}^{n-1} \right).
\]

(6.5.8)

Let us rewrite the scheme (6.5.4) in the following form

\[
u_{j}^{n+1} = kw_{-1}u_{j-1}^n + \left( 1 - kw_0 \right)u_j^n + kw_1u_{j+1}^n
\]

\[
+ \lambda k \left[ h \sum_{p \in P} \alpha_p u_{j+p}^n(\Gamma_\delta)_p - u_j^n \right] - k\tau_{h,k}^\epsilon,
\]

(6.5.9)

where \( \tau_{h,k}^\epsilon \) is the truncation error estimate in Proposition 6.5.1, which is \( \tau_{h,k}^\epsilon = O(k + h^2) + \lambda C\delta^2 \epsilon \). We want to estimate the global difference between the two numerical solution (6.5.6) and (6.5.9). Define

\[
e_{j}^{n+1} = v_{j}^{n+1} - u_{j}^{n+1}, \quad \text{and} \quad E^n = \sup_j | e_j^n |.
\]

We can prove the following result.

**Proposition 6.5.2** If \( \epsilon = O(h^4) \) and for the time step there holds a standard CFL condition, \( k = O(h^2) \), then \( E^n = O(h^2) \), for \( h \to 0 \).
Proof. Subtracting the equation (6.5.9) from the (6.5.6), we have
\[ e_j^{n+1} = kw_{-1}e_j^n + \left(1 - kw_0\right)e_j^n + kw_1e_{j+1}^{n+1} + \lambda k\sum_{p \in P_{in}} \alpha_p e_{j+p}^n (\Gamma_\delta)_p + h \sum_{p \in P_{out}} \alpha_p \left[ \tilde{v}_{j+p}^n - u_{j+p}^n \right] (\Gamma_\delta)_p - e_j^n + \delta \xi_{h,k}. \]

By (6.5.8) and (6.5.9), we obtain
\[ \tilde{v}_{j+p}^n - u_{j+p}^n = k \left( w_{-1} + \frac{\lambda D}{h^2} \right) e_{j+p-1} + \left[ 1 - k \left( w_0 + \frac{2\lambda D}{h^2} \right) \right] e_{j+p}^{n-1} \]
\[ + k \left( w_1 + \frac{\lambda D}{h^2} \right) e_{j+p+1}^{n-1} + \lambda k \left[ \frac{D}{h^2} \left( u_{j+p-1}^{n-1} - 2u_{j+p}^{n-1} + u_{j+p+1}^{n-1} \right) \right] \]
\[ - \left( h \sum_{p \in P} \alpha_p u_{j+p-1}^{n-1} (\Gamma_\delta)_p - u_{j+p}^{n-1} \right) + \delta \xi_{h,k}. \]

We have to estimate the difference between the central second order finite difference approximation and the integral approximation.

For every \( p \in P \) fixed
\[ u_{j+p} - u_j = (ph)u_x + \frac{(ph)^2}{2} u_{xx} + \frac{(ph)^3}{3!} u_{xxx} + \frac{(ph)^4}{4!} u_{xxxx} + \ldots \]

We call \( z_\epsilon = z_M = \delta \sqrt{-2 \log (\epsilon \delta \sqrt{2\pi})} \) as described in Subsection 6.3.1, and, for the compound rule (6.2.1), with \( \rho = 1 \), it exists a point \( \xi \in [-z_\epsilon, z_\epsilon] \) such that
\[ h \sum_{p \in P} \alpha_p \epsilon^3 (\Gamma_\delta)_p = \int_{-z_\epsilon}^{z_\epsilon} z^3 \Gamma_\delta(z) dz + \frac{h^2}{6} z_\epsilon^2 \left[ \frac{d^2}{dz^2} (z^3 \Gamma_\delta(z)) \right] \bigg|_{z=\xi}. \]

Then,
\[ h \sum_{p \in P} \alpha_p (u_{j+p} - u_j) (\Gamma_\delta)_p = \frac{1}{2} \left. \frac{d^2}{dx^2} u(x) \right|_{x=x_j} \int_{-z_\epsilon}^{z_\epsilon} z^2 \Gamma_\delta(z) dz + \frac{h^2}{6} z_\epsilon R_j. \]

This yields
\[ D \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} - h \sum_{p \in P} \alpha_p (u_{j+p} - u_j) (\Gamma_\delta)_p \]
\[ = \frac{1}{2} \left. \frac{d^2}{dx^2} u(x) \right|_{x=x_j} \left( 2D - \int_{-z_\epsilon}^{z_\epsilon} z^2 \Gamma_\delta(z) dz \right) - \frac{h^2}{6} z_\epsilon R_j. \]

Now, we choose the diffusion coefficient \( D \) such that
\[ D = \frac{1}{2} \int_{-z_\epsilon}^{z_\epsilon} z^2 \Gamma_\delta(z) dz. \]
Under the CFL condition (6.5.5) and for
\[ 1 - kw_0 - k \frac{2\lambda D}{h^2} \geq 0, \quad (6.5.13) \]
from (6.5.10) and (6.5.11), we have
\[ E^{n+1} \leq \left[ 1 - kc + \lambda kh \sum_{p \in P_{in}} (\Gamma_\delta)_p - \lambda k \right] E^n + \left[ \lambda k \left( 1 - kc \right) h \sum_{p \in P_{out}} \alpha_p (\Gamma_\delta)_p \right] E^{n-1} \]
\[ + \lambda k^2 h^2 \frac{z}{6} \sum_{p \in P_{out}} \alpha_p | R_{j+p} | (\Gamma_\delta)_p + \lambda k^2 \sum_{p \in P_{out}} \alpha_p (\Gamma_\delta)_p + k | \tau_{h,k}^\epsilon |. \]
This is, for some coefficients \( A, B \) and \( C \),
\[ E^{n+1} \leq AE^n + BE^{n-1} + C. \quad (6.5.14) \]
As a consequence, for any \( \frac{n+1}{2} \leq m < n \) we have
\[ E^{n+1} \leq E^{n-2m+1} \sum_{k=0}^{m} \binom{m}{k} (AE)^k B^{m-k} + C \sum_{k=0}^{m-1} (A + B)^k. \]
When \( n - 2m + 1 = 0, E^0 = 0 \) it yields
\[ E^{n+1} \leq C \frac{1 - (A + B)^{\frac{n+1}{2}}}{1 - (A + B)}. \quad (6.5.15) \]
Now, we have
\[ h \sum_{p \in P_{in}, P_{out}} (\Gamma_\delta)_p \leq 2z \max_\delta \Gamma_\delta = 2z \Gamma \text{ and } | R_j | \leq R. \]
The CFL condition (6.5.13) gives \( k = O(h^2) \) and \( \tau_{h,k}^\epsilon = O(h^2 + \epsilon) \). Then, for \( N = T/k \), the global error (6.5.15) is estimated by
\[ E^N \leq \left( 1 - \frac{T}{N} c - 2 \frac{T^2}{N^2} \lambda z \Gamma \right)^{N/2} g(h, \epsilon), \]
where
\[ g(h, \epsilon) = O\left( \frac{h^4 z^2 + h^4 \epsilon + h^2 z_e h^2 + h^2 + \epsilon}{1 + h^2 z_e} \right). \]
As \( h \) and \( k \) go to zero, we obtain
\[ \lim_{N \to \infty} (A + B)^{N/2} = \left( 1 - \frac{T}{N} c - 2 \frac{T^2}{N^2} \lambda z \Gamma \right)^{N/2} = e \frac{T C}{2}. \]
Then, to get the rate of convergence as $h \to 0$, we observe that the minimal value of the function $g(\cdot, \epsilon)$ is reached for $\epsilon = O(h^4)$.

Therefore, the conclusion follows, since

$$
\lim_{N \to \infty} E_N \leq e^{\frac{2\pi}{\sqrt{2}}} g(h, h^4)
\leq e^{\frac{2\pi}{\sqrt{2}}} \left( h^4 \log(-h^4) + h^4 \sqrt{\log(-h^4)} + h^6 \sqrt{\log(-h^4)} + h^2 + h^4 \right) = O(h^2).
$$

Remark 6.5.3 We point out that, for the Gaussian probability density (1.2.15) we have

$$
\bar{k} = E(\eta - 1) = \exp \left( \frac{\delta^2}{2} \right) - 1 \approx \frac{\delta^2}{2} + O(\delta^4), \quad \delta \ll 1,
$$

then, solving the approximated problem (6.4.2) in $\Omega^C$ is just solving the Black-Scholes equation with coefficients

$$
a = \frac{\sigma^2}{2} - r + \lambda \bar{k} \approx \frac{\sigma^2}{2} - r + \lambda \frac{\delta^2}{2}, \quad b = \frac{\sigma^2}{2} + \lambda \frac{\delta^2}{2}.
$$

Even if the scheme (6.5.6) needs for a CFL condition and its convergence in time is only first order accurate, we shall see in Subsection 6.6.1 that it is of simple practice application and computationally fast. It is easy to obtain a scheme which is second order in time, by applying the SSP (Strong Stability Preserving) Runge-Kutta technique, as in [30] and references therein, but we observe no real advantages for the total accuracy at least for the second order case. Then in what follows, we use scheme (6.5.6).

### 6.6 Examples and Numerical tests

In this section we compute the order $\gamma$ of the error in the following form

$$
\gamma = \log_2 \left( \frac{e_1}{e_2} \right),
$$

with

$$
e_p = \frac{\| u(h_p, T) - u(2h_p, T) \|_{1, \infty}}{\| u(h_p, T) \|_{1, \infty}}, \quad p = 1, 2,
$$

where $u(h)$ denotes the numerical solution obtained with the space step discretization equal to $h$, under the discrete norm $l^1$ and $l^\infty$, respectively

$$
\| u(\cdot, T) \|_1 = h \sum_i | u(x_i, T) |, \quad \| u(\cdot, T) \|_\infty = \max_i | u(x_i, T) |.
$$

If not specified, in tables that follow we give the average convergence order.
6.6.1 European option

Let us consider the problem of pricing an European option according to the problem (6.5.1). As we showed in Subsection 6.5.1, we solve the integro-differential equation on the numerical domain $\bar{\Omega}$. We apply the second order scheme (6.5.6) under the CFL condition

$$h \leq \frac{2b}{a}, \quad k \leq \min \left( \frac{h^2}{2b + 2\lambda D + ch^2}, \frac{h^2}{2b + ch^2 + \lambda h^2} \right),$$

(6.6.3)

with $D$ given by (6.5.12), $D = \lambda \delta^2 / 2$.

Let us fix the parameters as follows: $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\delta = 0.2$ and $T = 1$. In Figure 6.1, we present the value of the option given respectively by the jump-diffusion model (dotted-solid curve), the pure diffusion model ($\lambda = 0$) (solid curve), and the payoff value (dotted curve). The differences between the two models is clear: the value in the jump diffusion model is larger than the one in the pure diffusion setting in a neighborhood of the exercise price, according to the theoretical results in [50].

In Figure 6.2, we show the variation of the solution according to the jump intensity $\lambda$. We compare the solution with $\lambda = 0.5$ (○), $\lambda = 2$ (+) and $\lambda = 8$ (−) and we observe that the solutions increase with $\lambda$. This is what we expect from the model, because as the intensity of the jump increases the risk of the investment increases and consequently the price of the derivative needs to be higher.

In Tables $T2$ we show the $l^\infty$ errors and the convergence order (6.6.1) for the European call option initial data. This experimentally confirms that the scheme is second order accurate.

Since we might be interested in obtaining the value $u$ of the option for a given stock price $S^* = \exp(x^*)$, we fix $x^* = \ln(100)$ and $\sigma = 0.2$, $\delta = 0.8$, $\lambda = 0.1$. We compute the "exact" option price $P$ by the analytical solution given in [50]. Then, in Table $T3$ we show the convergence order,

$$\tilde{\gamma} = \log_2 \left( \frac{|u(h; x^*, T) - P|}{|u(\frac{h}{2}; x^*, T) - P|} \right),$$

(6.6.4)

where $u(h; x^*, T)$ is the numerical solution of (6.5.6) with space step $h$, valued in $x^*$ at time $T$. We stress out that we construct the interval $\Omega$ centered on $x^*$, then the values on the Table $T3$ are few influenced by the boundary error.

Table $T4$ shows CPU times on a 1,6 GHz Pentium IV PC for various number of space steps. Although the scheme (6.5.6) is of explicit type, it is computationally fast.
Examples and Numerical tests

Table T2: Example 6.6.1, $l^\infty$ errors and convergence orders of the European call option computed using the scheme (6.5.6). The process parameters are $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.8$. The number of time steps is given by the CFL condition (6.6.3).

<table>
<thead>
<tr>
<th>N</th>
<th>$T = 0.1$</th>
<th>$T = 1$</th>
<th>$T = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>129</td>
<td>0.000262</td>
<td>0.000778</td>
<td>0.002118</td>
</tr>
<tr>
<td>257</td>
<td>0.000123</td>
<td>0.000122</td>
<td>0.000645</td>
</tr>
<tr>
<td>513</td>
<td>0.000044</td>
<td>0.000036</td>
<td>0.000038</td>
</tr>
<tr>
<td>1025</td>
<td>0.000011</td>
<td>0.000016</td>
<td>0.000130</td>
</tr>
<tr>
<td>2049</td>
<td>0.000000</td>
<td>0.000008</td>
<td>0.000003</td>
</tr>
</tbody>
</table>

Convergence-order $T = 2.40096$, $T = 1.663561$, $T = 2.379236$

Table T3: Example 6.6.1, convergence orders (6.6.4) with respect to the analytical solution, see [50], of the European put and call option prices of Merton model computed using the explicit scheme (6.5.6). The process parameters are $E = 100$, $r = 0.05$, $\lambda = 0.1$, $\delta = 0.8$, $x = \ln(100)$. The number of time steps is given by the CFL condition (6.6.3).

<table>
<thead>
<tr>
<th>N</th>
<th>PUT</th>
<th>CALL</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1.562877</td>
<td>1.876357</td>
</tr>
<tr>
<td>128</td>
<td>2.356042</td>
<td>2.826212</td>
</tr>
<tr>
<td>256</td>
<td>2.572710</td>
<td>3.075204</td>
</tr>
<tr>
<td>512</td>
<td>2.625636</td>
<td>3.128717</td>
</tr>
<tr>
<td>1024</td>
<td>2.628921</td>
<td>3.122007</td>
</tr>
</tbody>
</table>

Analytical solution: 2.633642, 3.132394

Convergence-order γ: 1.956285, 2.916243

Table T4: Example 6.6.1, CPU times on 1.6 GHz Pentium IV PC for the scheme (6.5.6) when $T = 1$ and the number of time steps is given by the CFL condition (6.6.3). The CPU times for 64 and 128 nodes are not available.

<table>
<thead>
<tr>
<th>N</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU time (seconds)</td>
<td>N.A.</td>
<td>N.A.</td>
<td>0.02s</td>
<td>0.16s</td>
<td>4.4s</td>
</tr>
</tbody>
</table>

6.6.2 A two-dimensional example

In this section we present an operator splitting method for the two-dimensional degenerate equation (1.2.18). For an extensive description of operator splitting methods, we refer to the paper [40].

The main difficulty is given by the presence of a hyperbolic direction $y$. For simplicity, we set $r = 0$ and we write the equation in the short form

$$\partial_t u(x, y, t) + Du(x, y, t) = \lambda J u(x, y, t),$$
where
\[ Du = -b\delta^2_x u + a_1 \partial_x u + a_2 \partial_y u, \]
\[ Ju = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x + \xi, y + \eta, t) \Gamma(\xi, \eta) d\xi d\eta - u(x, y, t), \]
\[ \Gamma(\xi, \eta) = \frac{1}{2\pi \delta^2} \exp \left( -\frac{1}{2\delta^2} (\xi^2 + \eta^2) \right), \]
and \( b, a_1 \) and \( a_2 \) are constants.

The operator splitting method can be summarized as follows: let \( v^{n+1} = Dv^n \) be the numerical solution of
\[ \partial_t v(x, y, t) + Dv(x, y, t) = 0, \quad (6.6.5) \]
and let \( w^{n+1} = Jw^n \) be the numerical solution of
\[ \partial_t w(x, y, t) = \lambda Jw(x, y, t). \quad (6.6.6) \]

Then the operator splitting is based on the following approximation
\[ u^{n+1} = [JD]u^n. \]

To approximate the differential part (6.6.5), we shall apply an ADI method that combine Crank-Nicholson scheme in the two directions. To approximate the integral part (6.6.6), we shall apply the Euler rule for the time discretization and the compound Simpson’s product rule to the two-dimensional integral.

Let us define the two following discrete operators,
\[ D_x = \left[ k + \frac{\alpha_1 \delta_x}{2} - \frac{\beta \delta_{xx}}{2} \right], \quad D_y = \left[ k \alpha_2 \delta_y \right], \]
where
\[ \alpha_1 = \frac{a_1}{2h_1}, \quad \alpha_2 = \frac{a_2}{h_2}, \quad \beta = \frac{b}{h_1^2}, \quad \delta_{xx} u_{i,j} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \]
and
\[ \delta_y u_{i,j} = \begin{cases} (u_{i,j} - u_{i,j-1}) & a_2 \geq 0 \\ (u_{i,j+1} - u_{i,j}) & a_2 < 0 \end{cases} \]

We observe that we are using a central finite difference scheme for the \( x \)-direction and an upwind approximation for the degenerate one. We can now write the complete scheme. To calculate the numerical solution \( u^{n+1} \) from \( u^n \) we have to solve the following two steps:
1. We compute the values \( \tilde{u}_{i,j}^{n+1} \), for \( i,j = 0, \ldots, N \), by solving two tridiagonal systems,

\[
\begin{align*}
(I + D_x)\tilde{u}_{i,j}^{n+1} &= (I - D_y)u^n \\
(I + D_y)\tilde{u}_{i,j}^{n+1} &= (I - D_x)\tilde{u}_{i,j}^{n+1} + \frac{1}{2}
\end{align*}
\]  

(6.6.7)

where \( I \) denotes the identity matrix.

2. We obtain the solution \( u_{i,j}^{n+1} \), for \( i,j = 0, \ldots, N \), by the expression

\[
u_{i,j}^{n+1} = (1 - \lambda k)\tilde{u}_{i,j}^{n+1} + \lambda kh_1h_2 \sum_{l,m} \alpha_l \alpha_m \tilde{u}_{i+l,j+m}^{n+1} \Gamma_{l,m}.
\]  

(6.6.8)

We consider the following example,

\[
\begin{align*}
\partial_t u(x,y,t) + Du(x,y,t) &= \lambda Ju(x,y,t) \\
u(x,y,0) &= u_0(x,y)
\end{align*}
\]  

(6.6.9)

where we fix the parameters \( \lambda = 1, b = 1, a_1 = -a_2 = 0.5 \) and the \( x \) space discretization \( h_1 \) equal to the \( y \) space discretization \( h_2 \).

In Table T5, we show the \( \gamma \) order (6.6.1) under the norm \( l^1 \) of the scheme (6.6.7)-(6.6.8) applied to the problem (6.6.9) with a regular initial data \( u_0(x,y) = \sin(\pi(x+y)) \). We point out that we have chosen an upwind approximation to deal with the pure hyperbolic direction \( y \), then the scheme is at most first order accurate, as well verified in Table T5.

<table>
<thead>
<tr>
<th>( h_1 = h_2 )</th>
<th>( \delta = 10^{-4} )</th>
<th>( \delta = 10^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>0.099739</td>
<td>0.089381</td>
</tr>
<tr>
<td>0.0125</td>
<td>1.308349</td>
<td>0.040273</td>
</tr>
<tr>
<td>0.00625</td>
<td>0.557111</td>
<td>0.027372</td>
</tr>
<tr>
<td>0.003125</td>
<td>0.9288881</td>
<td>0.014378</td>
</tr>
</tbody>
</table>

Table T5: Convergence order \( \gamma \), defined in (6.6.1), and errors, for the solution of the problem (6.6.9) with \( u_0(x,y) = \sin(\pi(x+y)) \), \( b = 1, a_1 = -a_2 = 0.5 \).

### 6.6.3 The nonlinear case

As we have already seen in Section 1.2.3, the option pricing in large investor economy leads to a quasilinear differential problem. From equation (1.2.16), by the standard change of variable \( x = \log S \), we get the following general equation

\[
u_t + \mathcal{L}_{Iu}u = H(x,t,u,Iu,Du),
\]
where $\mathcal{L}_T$ is a linear degenerate elliptic integro-differential operator and $H$ is a nonlinear integro-differential Hamilton-Jacobi operator.

The numerical approximation of Hamilton-Jacobi equations have been intensively studied, both for first and second order equations. We refer again to [24, 11] for classical results and to [51, 47, 44] for recent developments of high order accurate schemes, such as ENO, WENO, and central schemes.

Let us introduce some standard notations:

$$u^\pm = \Delta^\pm u_j = \frac{\pm(u_{j\pm 1}-u_j)}{h}, \quad \Delta^2 u_j = \frac{u_{j+1}-2u_j+u_{j-1}}{h^2}, \quad \hat{H}(u^+, u^-),$$

where $\hat{H}$ is a Lipschitz continuous numerical flux, which is monotone and consistent with $H$ [24], i.e.:

$$\hat{H}(p, p) = H(p).$$

Monotonicity here means that $\hat{H}$ is non-increasing in its first argument and non-decreasing in the other one. Two of the most useful admissible numerical fluxes are the local Lax-Friedrichs (LLF) flux and the Godunov flux, [51].

**Example 6.6.1** [Large institutional investor]. Let us consider the Merton model for the large investor economy. As we have seen in the Example 1.2.9, the interest rate $r$ depends on the wealth $\xi$ invested in stocks and the price function solves the quasi-linear final value problem (1.2.16). In the specific case of the large institutional investor, the interest rate decreases when too many wealth is invested in bonds, according to the law $r(S, t, \xi) = R(S, t)f(\xi)$ where $f$ is a positive continuous function such that, for a given wealth $\xi_0 \geq 0$ fixed, $f(\xi) = 1$ as $\xi \leq \xi_0$ and $f$ is decreasing as $\xi > \xi_0$, but $f(\xi)\xi$ non-decreasing. A good prototype of such type function $f$ is given by

$$f(\xi) = \begin{cases} 1, & \xi \leq \xi_0 \\ \alpha + \beta\xi_0^{-\gamma}, & \xi > \xi_0, \end{cases}$$

for all $\alpha$, $\beta$ and $\gamma$ such that $\alpha$, $\beta > 0$, $0 < \gamma \leq 1$ and $\alpha + \beta\xi_0^{-\gamma+1} = 1$. We select,

$$\gamma = \frac{1}{2}, \quad \beta = \frac{1}{2\sqrt{\xi_0}} \Rightarrow \alpha = \frac{1}{2},$$

and we fix constant the interest rate $R(S, t) = R$.

We want to solve the following one dimensional quasi-linear problem,

$$\begin{cases} u_t - bu_{xx} + au_x + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{\lambda}\right) F_t(z) \frac{dz}{\sqrt{\lambda}} = T u, \\ u(x, 0) = \psi(x), \end{cases} \quad (6.6.10)$$

where $a = (\lambda\tilde{k} + \sigma^2/2)$, $b = \sigma^2/2$ and the non linear term $H$ is given by

$$H(u, p, q) = Rf\left(u - \tilde{a}u_x - \tilde{b}(q - u)\right) \left(u - \tilde{a}u_x - \tilde{b}(q - u)\right).$$
We stress out that the $H$ operator verify the general assumptions $\textbf{F1}$, $\textbf{F2}$, $\textbf{F3}$, given in Section 3.2, then the Cauchy problem (6.6.10) has a unique viscosity solution in the sense of Definition 3.2.5. Moreover, $H(\cdot, p, \cdot)$ is a decreasing monotone function, convex for $\xi > \xi_0$.

To discretize the equation (6.6.10) we approximate the nonl inear term by

$$
\hat{H}_J(u^n_j, u^+, u^-, \sum_{p \in P} \alpha_p u^n_{j+p}(\Gamma_\delta)_p) = H(u^n_j, \frac{u^+ + u^-}{2}, \sum_{p \in P} \alpha_p u^n_{j+p}(\Gamma_\delta)_p),
$$

This is of course a Lipschitz continuous numerical flux, monotone and consistent with $H(\cdot, p, \cdot)$. Applying the explicit scheme (6.5.4) for the linear part, we get the following approximation: for $j = j_-,...,j_+$,

$$
u_{j}^{n+1} = \nu_{j}^{n} - \frac{ak}{2h} \Delta_{-} \nu_{j}^{n} + \frac{bk}{h^2} \Delta \Delta \nu_{j}^{n} - \lambda k \nu_{j}^{n}$$

$$+ \lambda k \sum_{p \in P} \alpha_p u^n_{j+p}(\Gamma_\delta)_p - k \hat{H}_J(u^n_j, u^+, u^-, \sum_{p \in P} \alpha_p u^n_{j+p}(\Gamma_\delta)_p).$$

The scheme verifies the general convergence result (6.1.2) under the following CFL condition,

$$
h \leq \frac{2b}{a} \frac{2bk}{h^2} + \lambda k + k \max_{u} \left[ \frac{dH}{du}(u, \cdot, \cdot) \right] - k \min_{q} \left[ \frac{dH}{dq}(\cdot, \cdot, q) \right] \leq 1,$$

As it has been done for the linear problem (6.5.1), on the numerical boundary domain $\Omega^C$ we approximate the integral term $\mathcal{T}u$ in (6.6.10) by the diffusive one $Du_{xx}$ and we solve the following equation,

$$
u_t - b \nu_{xx} + a \nu_x + H(u, u_x, Du_{xx}) = \lambda Du_{xx}, \quad (x, t) \in \Omega^C \times (0, T],$$

under the condition

$$
\frac{2bk}{h^2} + \frac{2\lambda Dk}{h^2} + k \max_{u} \left[ \frac{dH}{du}(u, \cdot, \cdot) \right] - Dk \min_{q} \left[ \frac{dH}{dq}(\cdot, \cdot, q) \right] \leq 1.
$$

We fix $\hat{a} = a$, $\hat{b} = b$, the parameters $E = 100$, $R = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.4$ and the initial data $\psi(x) = (e^x - E)_+$ as the call option payoff function.

In Tables T7 we show the $l^\infty$ errors and the convergence order (6.6.1) for $\xi_0 = 10^2$. This experimentally shows that the scheme is second order accurate.

Figure 6.3 shows the call option payoff function compared with the solution of (6.6.10) at time $T = 1$, with $\xi_0 = 10^2$ fixed.
<table>
<thead>
<tr>
<th>N</th>
<th>$T = 0.5$</th>
<th>$T = 1$</th>
<th>$T = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>0.007924</td>
<td>0.013589</td>
<td>0.032857</td>
</tr>
<tr>
<td>129</td>
<td>0.000850</td>
<td>0.002836</td>
<td>0.007711</td>
</tr>
<tr>
<td>257</td>
<td>0.000610</td>
<td>0.000072</td>
<td>0.000522</td>
</tr>
<tr>
<td>513</td>
<td>0.000071</td>
<td>0.000056</td>
<td>0.000052</td>
</tr>
<tr>
<td>1025</td>
<td>0.000033</td>
<td>0.000068</td>
<td>0.000013</td>
</tr>
</tbody>
</table>

| Convergence-order | 1.977426 | 1.911010 | 2.839315 |

Table T7: Example 6.6.1, $l^\infty$ errors and convergence orders of the European call option computed using the scheme (6.6.11). The process parameters are $\xi_0 = 10^2$, $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.4$. The number of time steps is given by the CFL condition (6.6.12).
Figure 6.1: Subsection 6.6.1, jump-diffusion model (○), pure diffusion model (−) and payoff value (⋯), with Simpson compound rule and $h = 0.05$.

Figure 6.2: Subsection 6.6.1, jump-diffusion model with different values of jump intensity, $\lambda = 0.5$ (○), $\lambda = 2$ (+), $\lambda = 8$ (−).
Figure 6.3: Example 6.6.1, we show the call option payoff function compared with the solution of (6.6.10) ($\times$) at time $T = 1$, with $\xi_0 = 10^2$ fixed.
Chapter 7

Implicit-Explicit Schemes

In this chapter we focus our attention on the numerical approximation of the PIDE coming from the jump-diffusion model with constant coefficients.

\[
\begin{align*}
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} &= b \frac{\partial^2 u}{\partial x^2} - cu + \lambda \left( \int_{-\infty}^{\infty} u(x + z, t) \Gamma_\delta(z) dz - u(x, t) \right), \\
u(x, 0) &= \psi(x),
\end{align*}
\]  

(7.0.1)

where $\Gamma_\delta$ is defined in (1.2.15) and

\[
a = -(r - \lambda \bar{k} - \frac{1}{2} \sigma^2), \quad b = \frac{1}{2} \sigma^2, \quad c = r, \quad \bar{k} = \mathbb{E}(\eta - 1),
\]  

(7.0.2)

In the previous chapter we have shown a convergence theory for monotone explicit schemes for a general class of integro-differential Cauchy problems and we have shown an explicit finite-differences scheme to solve different integro-differential models. This rigorous approach requires a diffusive CFL conditions, namely $\Delta t \sim \Delta x^2$.

To use smaller time step size, we need to involve some implicit mechanism in the scheme. Unfortunately, due to the non-local nature of the integral term, a time implicit differencing scheme, unconditionally stable, is not practically feasible, and implicit schemes based on the FFT or wavelet basis fails to keep the monotonicity properties.

To overcome these problems, in this chapter we apply the Implicit-Explicit (IMEX) technique, which has been introduced for time dependent partial-differential equations that involve terms of different types. IMEX schemes have been widely used, especially in conjunction with spectral methods [43, 17], for the time integration of spatially discretized PDEs of diffusion-convection type. Typically, for these problems an implicit scheme is used for the diffusion term and an explicit scheme is used for the convection term. Some schemes of this type were proposed and analyzed as far back as the late 1970’s. Instances of this methods have been successfully applied to the incompressible Navier-Stokes equations [41] and in environmental modeling studies [62]. We refer to [54, 52, 53] for recent developments.
First, we shall show the behavior of an IMEX approximation, studying the order and the stability properties of these schemes. The IMEX schemes are high order schemes under weak stability condition. We first focus our attention on the pure differential case and we analyze three particular approximation, a second order one, the Midpoint-122 rule and two third order schemes, called G-233 and SSP3-433. For these approximation we give the CFL condition, showing that the Midpoint-122 is stable under the restriction $k = O(h^{4/3})$, while the G-233 and SSP3-433 are stable under the weak hyperbolic condition $k = O(h)$. We then, prove that adding the integral term to the problem does not change the goodness of such approximation. If the integral term is bounded the stability of the IMEX approximation applied to the pure differential problem guarantees the stability when considering the integro-differential equation.

Finally let us recall that two supplementary problems arise when discretizing the PIDE: the infinite domain and the non-local nature of the integral term. Here we generalize the approach given in the previous chapter (see Section 6.5.2) to deal with the truncation of the domain and to implement numerical boundary conditions.

The chapter is organized as follows: in Section 7.1, we introduce the general structure of IMEX Runge-Kutta schemes: sub-section 7.1.2 is devoted to the consistency analysis, while sub-section 7.1.3 contains the analysis of the stability properties of such approximation. We also analyze in detail a second order and a third order IMEX approximations. We shall establish the optimal CFL conditions to apply to these particular schemes. Section 7.2 is devoted to the description of the numerical implementation; we give the algorithm to solve the PIDE coming from the Merton’s model [50], with IMEX schemes combined with the artificial boundary conditions proposed in Section 6.5.2. We then evaluate the computational costs. In the last Section we apply the schemes presented in Section 7.2 to find numerically the value of European Vanilla options in the Merton’s model.

Finally, let us observe that our methods can be directly used for more general nonlinear problems, possibly in several space dimensions, although to establish the rigorous nonlinear stability of these schemes seems to be a difficult task to be addressed in a future work.

### 7.1 The IMEX Runge-Kutta schemes

Let us consider the general linear parabolic integro-differential equation in $\mathbb{R} \times (0, T)$,

$$u_t(x, t) - H(Du(x, t), Iu(x, t)) = G(u(x, t), D^2u(x, t)), \quad (7.1.1)$$

with

$$Iu = \lambda \left( \int_{-\infty}^{\infty} u(x + z, t) \Gamma(z) dz - u(x, t) \right), \quad (7.1.2)$$

where $G$ is a linear elliptic operator and $H$ is a linear first order operator, depending on an integral term $Iu$. 


We define a numerical grid in $\mathbb{R} \times (0,T)$ using the following notation: $h$ is the spatial grid size, $k$ is the time grid size, $(x_m,t_n) = (mh,nk)$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, are the grid points, $v_m^n$ is the value of the function $v$, defined on the grid or defined for continuously varying $(x,t)$, at the grid point $(x_m,t_n)$ and $\hat{v}$ is the vector of $v$ values, $(v_m^n)_{m,n}$ for $m$ and $n$ varying on a subset of $\mathbb{Z}$ and $\mathbb{N}$ respectively.

As a first step in the discretization of (7.1.1), the operators $\mathcal{H}$ and $\mathcal{G}$ are replaced by an appropriate spatial discretization (central finite differences, finite-element, finite volume or spectral methods) coupled with an integral approximation (quadrature formulas or FFT technique) for the term $Iu$. We shall denote by $\mathcal{H}$ and $\mathcal{G}$, respectively, the discretization of $\mathcal{H}$ and $\mathcal{G}$. We shall write

$$\int_{-\infty}^{+\infty} u(x_j + z,t)\Gamma(z)dz - u(x_j,t) \approx h \sum_{p \in P} \alpha_p u(x_{j+p},t)(\Gamma_\delta)_p - u(x_j,t) = J(u(t)).$$

(7.1.3)

The resulting semidiscrete approximation, called the method of lines, leads to a large system of ODEs of the form

$$\dot{u}(t) = H(u) + G(u),$$

(7.1.4)

where $G(u)$ depends on the $x$-approximation of second order term, $\mathcal{G}$. We shall suppose that the function $H$ is linear and we shall write

$$H(u) = F(u) + J(u),$$

(7.1.5)

where $F(u)$ depends on the approximation of the first order term $Du$, and $J(u)$ is defined by (7.1.3).

In Section 6.5.1, an explicit scheme was investigated, which was only first order in time, and second order in space. Since $k = O(h^2)$, then a second order scheme under parabolic CFL is obtained. Due to the non-local nature of the integral term, an implicit time differencing scheme, unconditionally stable, is not practically feasible. The convolution integral leads to a dense system of equations which is hard to solve. Hence, a natural idea is to study asymmetric treatments (implicit-explicit) of the differential and the integral part to obtain high order accuracy in time under weaker stability conditions. To achieve this goal, it is natural to make use of Implicit-Explicit (IMEX) time discretization schemes, widely used for time integration of spatially discretized PDEs of diffusion-convection type.

Therefore, we want to integrate explicitly the term $H(u)$, while $G(u)$ is considered as a stiff term, which should be integrated implicitly to avoid excessively small time steps. A general IMEX-DIRK Runge-Kutta scheme for system (7.1.4) takes the
following form:

\[
\begin{align*}
  u^{(i)} &= u^n + k \sum_{j=1}^{i-1} \hat{a}_{ij} H(u^{(j)}) + k \sum_{j=1}^{i} a_{ij} G(u^{(j)}), \quad i = 1, \ldots, \nu \\
  u^{n+1} &= u^n + k \sum_{i=1}^{\nu} \hat{\omega}_i H(u^{(i)}) + k \sum_{i=1}^{\nu} \omega_i G(u^{(i)}).
\end{align*}
\] (7.1.6)

The matrices \( \tilde{A} = (\tilde{a}_{ij}) \) and \( A = (a_{ij}) \) are \( \nu \times \nu \) matrices such that the resulting scheme is explicit in \( H \) and implicit in \( G \). DIRK formulation requires \( a_{ij} = 0 \) for \( j > i \).

Following the IMEX formalism we shall use the notation \( \text{Name}(s, \nu, p) \) to identify a scheme, where \( s \) is the number of stages of the implicit scheme, \( \nu \) is the number of explicit scheme stages and \( p \) is the combined order of the scheme. We refer to [53], [54], [42] and references therein for considerations on the generalization of the concept of order conditions and absolute stability for Runge-Kutta schemes to IMEX Runge-Kutta.

Let us focus our attention on the linear PIDE equation (7.0.1). We first look for a solution which is a Fourier mode of the form \( u(x, t) = \exp(i\xi x)\hat{u}(t) \). The evolution equation for such mode is

\[
\hat{u}_t(t) = \hat{H}(\xi)\hat{u}(t) + \hat{G}(\xi)\hat{u}(t),
\] (7.1.7)

where for the linear equation (7.0.1), with \( r = 0 \) we have

\[
\hat{H}(\xi) = -ia\xi + \lambda\left( \int_{\mathbb{R}} e^{i\xi z} \Gamma(z) dz - 1 \right), \quad \hat{G}(\xi) = -b\xi^2.
\] (7.1.8)

We apply the scheme (7.1.6) to equation (7.1.7). Following [53], let \( e = (1, \ldots, 1)^T \) denote a column vector whose components are unitary. Set

\[
U = \begin{pmatrix} \hat{u}^{(1)} \\ \vdots \\ \hat{u}^{(\nu)} \end{pmatrix}, \quad U^n = e \cdot \hat{u}^n = \begin{pmatrix} \hat{u}^n \\ \vdots \\ \hat{u}^n \end{pmatrix},
\] (7.1.9)

\[
A = \tilde{G}(\xi)A, \quad \tilde{A} = \hat{H}(\xi)\tilde{A}.
\]

The scheme (7.1.6), applied to the scalar equation (7.1.7), can then be written as

\[
\begin{align*}
  U &= U^n + k\tilde{A}U + kAU \\
  \hat{u}^{n+1} &= \hat{u}^n + k\hat{H}(\xi)\hat{\omega}^T U + k\hat{G}(\xi)\omega^T U.
\end{align*}
\] (7.1.10)

Introducing the following notation

\[
z^{(E)} = k\hat{H}(\xi), \quad z^{(I)} = k\hat{G}(\xi),
\] (7.1.11)
solving the first equation for $U$ and substituting the solution into the second equation, one obtains
\[ \hat{u}^{n+1} = R(z^{(E)}, z^{(I)}) \hat{u}^n, \] (7.1.12)
where
\[ R(z^{(E)}, z^{(I)}) = 1 + (z^{(E)} \tilde{\omega}^T + z^{(I)} \omega^T)(I_{\nu} - z^{(E)} \tilde{A} - z^{(I)} A)^{-1} e. \] (7.1.13)
As showed in [42], some algebra gives
\[ R(z^{(E)}, z^{(I)}) = P(z^{(E)}, z^{(I)}) Q(z^{(I)}), \] (7.1.14)
where
\[ P(z^{(E)}, z^{(I)}) = \sum_{i=0}^{\nu} \sum_{j=0}^{\nu-1} p_{ij} (z^{(E)})^j (z^{(I)})^i, \] (7.1.15)
\[ Q(z^{(I)}) = 1 + \sum_{i=1}^{\nu-1} q_i (z^{(I)})^i, \]
for $p_{ij}, q_i$ depending on the matrix $\tilde{A}, A$ and on the coefficients $\tilde{\omega}, \omega$. In all cases $p_{00} = 1$ and $p_{\nu 0} = p_{\nu -10} = 0$ so that the stability function $R$ vanishes as $z^{(I)}$ tends toward infinity.

**Remark 7.1.1** Let us consider, as example, the standard centered finite difference approximation
\[ u_{xx} \approx \delta_{xx}^2 u(x) = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2}, \quad u_x \approx \delta_x u(x) = \frac{u_{m+1} - u_{m-1}}{2h}, \] (7.1.16)
for the equation (7.0.1) combined with the integral approximation (7.1.3). Using the Fourier inversion formula for $u(x_m, t)$ we have
\[ u(t, x_m) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{im\xi} \hat{u}(t, \xi) d\xi. \] (7.1.17)
Substituting this expression in the equation (7.0.1) we get the following scalar ODE:
\[ \dot{\hat{u}}(t) = H(\theta; \hbar) \hat{u} + G(\theta; \hbar) \hat{u}, \] (7.1.18)
where $-\pi \leq \theta = h\xi \leq \pi$ and
\[ H(\theta; \hbar) = F(\theta; \hbar) + J(\theta; \hbar), \quad G(\theta; \hbar) = \frac{2b}{h^2} (\cos \theta - 1), \]
\[ F(\theta; \hbar) = -i \frac{\alpha}{\hbar} \sin \theta, \quad J(\theta; \hbar) = \lambda \left( \int_{-\infty}^{+\infty} e^{iz\theta/h} \Gamma(z) dz - 1 \right). \] (7.1.19)
The IMEX-DIRK scheme (7.1.6), applied to the scalar equation (7.1.18), becomes
\[ \hat{u}^{n+1} = g(\theta; h, k)\hat{u}^n, \]  
(7.1.20)
then, from (7.1.12)-(7.1.13),
\[ g(\theta; h, k) = R(kH(\theta; h), kG(\theta; h)). \]

### 7.1.1 Examples in the unperturbed case

Here we present some schemes already in use for the classical advection-diffusion equations.

**Implicit-Explicit Midpoint**(1,2,2) The Midpoint(1,2,2) scheme is given by the general approximation (7.1.6) with \( s = 1, \nu = 2 \) and
\[ \tilde{A} = (\tilde{a}_{ij}) = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}, \quad A = (a_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}, \]  
(7.1.21)
\[ (\tilde{\omega}_1, \tilde{\omega}_2) = (0, 1), \quad (\omega_1, \omega_2) = (0, 1). \]

**A third-order combination** (2,3,3) A two-stage, third-order DIRK scheme, see [7], is given by the general approximation (7.1.6) with \( s = 2, \nu = 3 \) and, for \( \gamma = \frac{3 + \sqrt{3}}{6} \)
\[ \tilde{A} = (\tilde{a}_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ \gamma - 1 & 2 - 2\gamma & 0 \end{pmatrix}, \quad A = (a_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 1 - 2\gamma & \gamma \end{pmatrix}, \]  
(7.1.22)
\[ (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = (0, 1/2, 1/2), \quad (\omega_1, \omega_2, \omega_3) = (0, 1/2, 1/2). \]
We shall call this scheme with the short form ARS-233.

The **IMEX-SSP3**(4,3,3) **L-stable scheme** Another third order SSP (Strong Stability Preserving) scheme has been proposed by Pareschi and Russo, see [53] and references therein.
\[ \tilde{A} = (\tilde{a}_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 \end{pmatrix}, \quad A = (a_{ij}) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ -\alpha & \alpha & 0 & 0 \\ 0 & 1 - \alpha & \alpha & 0 \\ \beta & \eta & 1/2 - \beta - \eta - \alpha & \alpha \end{pmatrix}, \]  
(7.1.23)
\[ (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4) = (0, 1/6, 1/6, 2/3), \quad (\omega_1, \omega_2, \omega_3, \omega_4) = (0, 1/6, 1/6, 2/3). \]
\( \alpha = 0.24169426078821, \ \beta = 0.06042356519705, \ \eta = 0.1291528696059. \)

### 7.1.2 Consistency and order conditions

Let us consider again the approximation (7.1.6) in the linear setting. To get order conditions for the scheme we use symbol analysis, see Section 4.2.3. To have a discussion on the nonlinear case, see [42].

Let \( p(s, \xi) \) be the symbol of a general linear partial differential operator,

\[
P(s, \xi) = s + q(\xi),
\]

and let \( H \) and \( G \) be two given consistent space approximation of accuracy order \( O(h^\beta) \), i.e.

\[
H(h\xi; h) + G(h\xi; h) = -q_1(\xi) - q_2(\xi) + O(h^\beta) = -q(\xi) + O(h^\beta). \tag{7.1.24}
\]

As an example, see Section 6.5.1 where we show that the space approximation of the problem (7.0.1), defined in (7.1.16)-(7.1.19) where \( \theta = h\xi \) is accurate of second order, see [16].

**Proposition 7.1.2** Let be given a linear differential problem and let \( H \) and \( G \) be two consistent space approximation that verify (7.1.24). The scheme (7.1.10) is accurate of order \( O(h^\beta + k\alpha) \) if the two matrices \( \tilde{A} \) and \( A \) and the coefficients \( \tilde{\omega} = (\tilde{\omega}_1, ..., \tilde{\omega}_\nu)^T \) and \( \omega = (\omega_1, ..., \omega_\nu)^T \) verify

\[
\left( q_1(\xi)\tilde{\omega}^T + q_2(\xi)\omega^T \right) \left( \frac{d^{j-1}\Lambda(k)}{dk^{j-1}} \right)_{k=0} = (-1)^{j+1} \frac{\left( q_1(\xi) + q_2(\xi) \right)^j}{j} \text{ for } j = 1, ..., \alpha,
\]

where \( \Lambda(k) = (I_\nu + k\tilde{A}q_1(\xi) + kAq_2(\xi))^{-1}e. \)

**Proof.** By Theorem 4.2.5 we look for a symbol \( r_{k,h}(s, \xi) \) such that

\[
p_{k,h}(s, \xi) - r_{k,h}(s, \xi)p(s, \xi) = O(h^\beta + k\alpha). \tag{7.1.26}
\]

Substituting \( u_m^n = e^{sk}e^{ih\xi} \) in (7.1.10), we get the symbol \( p_{k,h}(s, \xi) \) of the difference scheme,

\[
p_{k,h}(s, \xi) = \frac{e^{sk} - 1}{k} - (H(h\xi; h)\tilde{\omega}^T + G(h\xi; h)\omega^T)\Lambda(k). \tag{7.1.27}
\]

By Taylor expansion centered on \( k = 0 \), the symbol (7.1.27) is given by

\[
p_{k,h}(s, \xi) = s + \frac{k}{2}s^2 + \frac{k^2}{3!}s^3 + ... + \frac{k^{\alpha-1}}{\alpha!}s^\alpha + O(k^\alpha)
\]

\[
+ \sum_{j=1}^{\alpha} k^{j-1} \left( q_1(\xi)\tilde{\omega}^T + q_2(\xi)\omega^T \right) \left( \frac{d^{j-1}\Lambda(k)}{dk^{j-1}} \right)_{k=0} + O(h^\beta). \tag{7.1.28}
\]
Relatively simple algebraic considerations shows that if we choose the symbol $r_{k,h}$ such that
\[
r_{k,h}(s,\xi) = 1 + \frac{b}{2}\left(s - q(\xi)\right) + \frac{k}{4!}\left(s^3 - q^3(\xi) - sq(\xi)\right) + \frac{k^3}{4!}\left(s^3 - q^3(\xi) + sq^2(\xi) - s^2q(\xi)\right) + \ldots + \frac{k^{\alpha-1}}{\alpha!}b_{\alpha-1}(s, -q(\xi)),
\]
where
\[
b_{\alpha-1}(s, -q(\xi)) = s^{\alpha-1} + s^{\alpha-2}(-q) + s^{\alpha-3}(-q)^2 + \ldots + s(-q)^{\alpha-2} + (-q)^{\alpha-1},
\]
the relation (7.1.26) is achieved if the $\alpha$ conditions (7.1.25) hold.

**7.1.3 Stability**

Relation (7.1.20) shows that advancing the solution of the scheme by one time step is equivalent to multiplying the Fourier transform of the solution by the *amplification factor* $g(\theta; h, k)$. The stability condition of the scheme (7.1.10) is then given by the general Theorem 4.2.1.

Let us now recall that $g(\theta; h, k) = R(kH(\theta; h), kG(\theta; h))$, with $R$ defined in (7.1.12) and $H, G$ in (7.1.19). For $z_J = kJ(\theta; h)$, $z = kF(\theta; h)$ and $z^{(E)} = z + z_J$, the expressions (7.1.14)-(7.1.15) give
\[
R(z^{(E)}, z^{(I)}) = R(z, z^{(I)}) + \frac{1}{Q(z^{(I)})} \sum_{i=0}^{\nu} \sum_{j=0}^{\nu-1} \frac{j!}{i!(j - i)!} p_{ij}(z)\left(z^{(I)}\right)^{i}(z^{(I)})^{j}.
\]
(7.1.29)

**Proposition 7.1.3** Let us suppose that $R(z, z^{(I)})$ verifies the stability condition
\[
| R(z, z^{(I)}) | \leq 1,
\]
for all $\theta$ and $0 < h \leq h_0$ and $k = O(h)$. If the additional term $z_J$ is $O(k)$ for all $\theta$ and $h$, then, the scheme (7.1.6) is stable for the integro-differential problem (7.0.1).

**Proof.** The additional term in expression (7.1.29) is a polynomial in $z_J$,
\[
\sum_{i=0}^{\nu} \sum_{j=0}^{\nu-1} \frac{j!}{i!(j - i)!} p_{ij}(z)\left(z^{(I)}\right)^{i}(z^{(I)})^{j}.
\]
(7.1.30)

When $l > 0$, the term in square bracket is uniformly bounded for the stability hypothesis on $R(z, z^{(I)})$. For $l = 0$ the all term (7.1.30) reduces to
\[
\frac{1}{Q(z^{(I)})} \sum_{i=0}^{\nu} p_{i0}(z^{(I)})^i = O(h) = O(k), \text{ for } i < \nu - 1,
\]
and we recall that $p_{\nu,0} = p_{\nu-1,0} = 0$. Since $z_J = O(k)$, the thesis is then achieved, in fact:
\[
| R(z^{(E)}, z^{(I)}) | \leq | R(z, z^{(I)}) | + Kk \leq 1 + Kk,
\]
for some constant $K$ independent of $\theta$, $h$ and $k$. 

\[\square\]
Let us now fix our attention to the pure advection-diffusion case, i.e. we set \( \lambda = 0 \) in (7.0.1). We then compute the function \( R \) of absolute stability with \( z^{(E)} = -ia\xi k \) and \( z^{(I)} = -b\xi^2 k \). We plot the regions \( R(\cdot, \cdot) \) of schemes (7.1.21)-(7.1.22)-(7.1.23), the border of such regions is given by the relation

\[
| R(-ia\xi k, -b\xi^2 k) | = 1.
\]

The plot is shown in Figure 7.1 in logarithmic scale. The stability region is located below the curve. We can observe that stability increases for high order schemes.

To implement a scheme we need an explicit law to choose the grid steps \( h \) and \( k \). Let then, the space approximation (7.1.16) be given. We fix in (7.1.19), \( A = ka/h \), \( B = 2bk/h^2 \) and \( \lambda = 0 \). Imposing \( y = \sin^2 (\theta/2) \), the condition (4.2.10) is equivalent to the following one,

\[
\forall y \in [0, 1], F(y; h, c) = \left| \sum_{i=0}^{\nu} \sum_{j=0}^{\nu-1} p_{ij} (-2)^{i+j} (iA)^j B^i (y)^{i+j/2} (1-y)^{j/2} \right| - (1 + 2\gamma By)^{\nu-1} \leq 0,
\]

(7.1.31)

where we fixed \( \gamma = a_{ii} > 0 \). Schemes with this assumption are called SDIRK (Singly DIRK) and this assumption is generally used for computational reasons.

When \( k = ch^{1+\alpha} \), the inequality (7.1.31) must hold for all small positive values of \( h \) uniformly in \( y \). We have

\[
A \sim h^\alpha, \quad B \sim h^{-(1-\alpha)},
\]

and we look for the value of the constant \( c \) and the smallest value of \( \alpha < 1 \) such that

\[
\lim_{h \to 0} F(y; h, c h^{1+\alpha}) \leq 0 \quad \forall y \in [0, 1].
\]

(7.1.32)

Actually the admissible value \( \alpha = 0 \) is the most interesting one.

**Examples**

In this subsection we shall consider some examples of IMEX approximations for the advection-diffusion problem (7.0.1), when \( \lambda = 0 \) and \( c = 0 \). We shall give the CFL condition for the examples we have considered in subsection 7.1.1, where \( a \) and \( b \) are the advection and the diffusion coefficient respectively.

**Implicit-Explicit Midpoint(1,2,2)** The scheme verifies the condition (7.1.25) for \( j = 1, 2 \), then it is second order in time.

Some algebra shows that the condition (7.1.32) gives, for \( \alpha \geq 1/3 \)

\[
f(y) \equiv -2b + 4 h^{-1+3\alpha} c^3 a^2 b^2 y^2 - 4h^{-1+3\alpha} c^3 a^2 b^2 y^3 \leq 0.
\]

(7.1.33)

The function \( f(y) \) achieves its maximum value \( \forall y \in [0, 1] \), on \( y = 2/3 \), which yields:

\[
f(2/3) \leq 0 \quad \Leftrightarrow \quad c \leq \frac{3h(\alpha-1/3)}{2(a^2 b)^{1/3}}.
\]

(7.1.34)
We choose the "optimal" value \( \alpha = 1/3 \) and the Midpoint scheme is stable for
\[
k \leq \frac{3}{2(a^2 b)^{1/3} h^{1/3}}. \tag{7.1.35}
\]

**ARS(2,3,3)** The scheme is third order in time, it verifies the condition (7.1.25) for \( j = 1, 2, 3 \).

Some algebra shows that the condition (7.1.32) with \( \alpha = 0 \) gives
\[
f(y) \equiv 16 c^3 b^4 y^3 (2 \gamma - 1)^2 (-16 c^2 y a^2 \gamma^2 + 16 c^2 y^2 a^2 \gamma^2 + 4 \gamma - 1) \geq 0.
\tag{7.1.36}
\]

The function \( f(y) \), assumes its minimum value \([0, 1] \) in \( y = 1/2 \), which yields:
\[
f(1/2) \geq 0 \iff c \leq \frac{\sqrt{4 \gamma - 1}}{2a \gamma}.
\tag{7.1.37}
\]

The scheme is stable for
\[
k \leq \frac{\sqrt{4 \gamma - 1} h}{2 \gamma a}.
\tag{7.1.38}
\]

**The IMEX-SSP3(4,3,3) L-stable scheme** The scheme is third order in time, it verifies the condition (7.1.25) for \( j = 1, 2, 3 \).

Some algebra shows that the condition (7.1.32) with \( \alpha = 0 \) gives
\[
f(y) \equiv 4 c^2 a^2 y^2 - 4 c^2 a^2 y + 3 \geq 0 \tag{7.1.39}
\]

It then yields:
\[
f(1/2) \geq 0 \iff c \leq \frac{\sqrt{3}}{a}.
\tag{7.1.40}
\]

The scheme is stable for
\[
k \leq \frac{\sqrt{3} h}{a}.
\tag{7.1.41}
\]

**The integro-differential models**

Let us now turn again to the integro-differential case (7.0.1). By the Proposition 7.1.3 we can use the previous results to implement the IMEX approximation to the integro-differential problem.

First, we fix the parameter \( b = a \) and we plot the border of the stability region given by the relation
\[
| R(k F(\theta; h) + \lambda k, k G(\theta; h)) | = 1,
\tag{7.1.42}
\]
in the plane \( h-k \), varying the parameter \( \lambda \). The plot is shown in Figure 7.2, 7.3, 7.4. In all cases the stability region is located below the curves. However, for small value
of the time step \( k \) (as we expect due to the Proposition 7.1.3), the CFL conditions (7.1.35), (7.1.38), (7.1.41) (represented by the straight line) guarantee stability.

When considering the problem (7.0.1) from a financial point of view, we point out that the jump intensity \( \lambda \) influences the convection parameter \( a \) (see the coefficients in (7.0.2)). The CFL conditions then modifies varying \( \lambda \) and we show in Figure 7.5, 7.6, 7.7 that the schemes considered are stable under the condition (7.1.35), (7.1.38), (7.1.41). Moreover, in Figure 7.5, we plot the curve \( k = h/a \). For small values of \( h \) this curve gives instability to the Midpoint-122 scheme.

In Figure 7.5, 7.6, 7.7, we fix the process parameters as \( r, \sigma, \delta \) and \( \lambda \) and we plot the stability region \( R \) for the schemes (7.1.21)-(7.1.22)-(7.1.23) respectively. Moreover, we plot the curves given by the CFL relations (7.1.35), (7.1.38), (7.1.41). These curves belong to the stability domain.

### 7.2 Numerical implementation

Let us consider the Merton’s model (7.0.1). To apply a numerical scheme, it is necessary to truncate the problem domain on one hand, and the integral domain on the other. Once we have found a given domain, still we need to use some approximation of the solution in a larger computational domain. Following Section 6.5.2, we call \( \Omega \subset \mathbb{R} \) the interval where we want to compute the numerical solution. We first truncate the integral domain of the problem. We choose a parameter \( \epsilon > 0 \) and select the bounded interval \( [-z_\epsilon, z_\epsilon] \) as the set of all points \( z \) that verify

\[ \Gamma_\delta(z) \geq \epsilon. \]  

(7.2.1)

By simple calculation we derive

\[ -z_\epsilon \leq z \leq z_\epsilon, \quad z_\epsilon = \sqrt{-2\delta^2 \log (\epsilon \delta \sqrt{2\pi})}. \]  

(7.2.2)

The truncation of the integral domain gives an error to the approximation of the problem. For a given function \( u(\cdot, t) \in \text{Lip}(\mathbb{R}) \) with constant \( L \), this error can be estimated by

\[
\left| \int_{-\infty}^{+\infty} [u(x+z, t) - u(x, t)] \Gamma_\delta(dz) - \int_{-z_\epsilon}^{+z_\epsilon} [u(x+z, t) - u(x, t)] \Gamma_\delta(dz) \right| \\
\leq L \left( \int_{-\infty}^{-z_\epsilon} |z| \Gamma_\delta(dz) + \int_{z_\epsilon}^{+\infty} |z| \Gamma_\delta(dz) \right) \\
= 2 \int_{z_\epsilon}^{+\infty} z \frac{1}{\sqrt{2\pi}\delta} \exp \left( -\frac{z^2}{2\delta^2} \right) dz = \frac{2\delta^2}{\sqrt{2\pi}} \exp \left( -\frac{z_\epsilon^2}{2\delta^2} \right) = 2\delta^2 \epsilon.
\]

We stress that the integral is a nonlocal term. For \( \Omega = [x_{\text{min}}, x_{\text{max}}] \), we solve problem (7.0.1) in a bigger numerical domain \( \bar{\Omega} = [x_{\text{min}} - z_\epsilon, x_{\text{max}} + z_\epsilon] \). Let \( h \) be the space step, we call \( M \) the total number of grid points and \( P \) the total number of points.
used for the integral approximation. The grid is then defined by the set of points 
\( \{ x_m, m = 0, \ldots, M-1 \} \), where 
\( x_0 = x_{\text{min}} - z \epsilon \) and 
\( x_{M-1} = x_{\text{max}} + z \epsilon \). We shall define 
by \( j_- \) and \( j_+ \) the two grid indices such that 
\[ x_{j_-} = x_{\text{min}} \quad \text{and} \quad x_{j_+} = x_{\text{max}}, \]
and by \( p \) the integer number \( p = [z \epsilon / h] \).

Let us set 
\( u^n_m = u(x_m, nk) \) and denote by \( I_h(u^n_m) \) the integral approximations 
\[ I_h(u^n_m) = h \sum_{l=-p}^{p} \alpha_l(\Gamma_\delta)u^n_{m-l}, \quad (7.2.3) \]
where \( \alpha_l \) are, for example, the weights of a quadrature rule.

Then, as we have defined our numerical problem on \( \Omega \), we need a limiting form 
for the solution \( u \) on the external set \( \Omega^C = \mathbb{R} \setminus \Omega \). As shown in Section 6.4, the 
integral term can be replaced (locally) by a diffusive term. We then approximate 
on the external set \( \Omega^C \) the problem (7.0.1) by the diffusive one
\[ v_t + av_x - bv_{xx} + cv = Dv_{xx}, \quad D = \frac{\lambda \delta^2}{2}. \quad (7.2.4) \]

Let us now consider the standard centered finite differences approximation (7.1.16) 
and the IMEX schemes (7.1.6). We shall use the following notation:

- for \( a \) and \( b \) be the constant coefficients of (7.0.1), where we are assuming \( r = 0 \), 
we set \( B = kb/h^2, \bar{B} = k(b + D)/h^2 \) and \( A = ak/2h \);

- for \( i = 1, \ldots, \nu \), we set
  \[ g^i_{-1} = -Ba_{ii} \quad g^i_0 = 1 + 2Ba_{ii} + \lambda ka_{ii} \quad g^i_1 = -Ba_{ii} \]
  \[ \bar{g}^i_{-1} = -\bar{B}a_{ii} \quad \bar{g}^i_0 = 1 + 2\bar{B}a_{ii} \quad \bar{g}^i_1 = -\bar{B}a_{ii}; \]

- for \( i, j = 1, \ldots, \nu \), we set
  \[ \beta^i_{-1} = A\tilde{a}_{ij} + Ba_{ij} \quad \beta^i_0 = -2Ba_{ij} - \lambda ka_{ij} \quad \beta^i_1 = A\tilde{a}_{ij} + Ba_{ij} \]
  \[ \bar{\beta}^i_{-1} = A\tilde{a}_{ij} + Ba_{ij} \quad \bar{\beta}^i_0 = -2\bar{B}a_{ij} \quad \bar{\beta}^i_1 = A\tilde{a}_{ij} + \bar{B}a_{ij}; \]

Each time step of the IMEX-DIRK scheme (7.1.6) consists on solving the two following steps:
(S1) Solve the following tridiagonal systems, for \( i = 1, \ldots, s, \)
\[
\begin{align*}
g_{i-1}^{(i)} u_{m-1}^{(i)} + g_{0}^{(i)} u_{m}^{(i)} + g_{1}^{(i)} u_{m+1}^{(i)} \\
= u_{m}^{n} + \sum_{j=1}^{\nu} \left[ \alpha_{j}^{(i)} u_{m-1}^{(j)} + \alpha_{0}^{(i)} u_{m}^{(j)} + \alpha_{1}^{(i)} u_{m+1}^{(j)} \right] \quad m = 0, \ldots, j - 1; \\
g_{i-1}^{(i)} u_{m-1}^{(i)} + g_{0}^{(i)} u_{m}^{(i)} + g_{1}^{(i)} u_{m+1}^{(i)} \\
= u_{m}^{n} + \sum_{j=1}^{\nu} \left[ \alpha_{j}^{(i)} u_{m-1}^{(j)} + \alpha_{0}^{(i)} u_{m}^{(j)} + \alpha_{1}^{(i)} u_{m+1}^{(j)} + k \lambda \varphi_{ij} \mathcal{I}_{h}(u_{m}^{(j)}) \right] \quad m = j, \ldots, j + 1; \\
\end{align*}
\]
(7.2.5)

(S2) Calculate the vector values \( (u_{m+1}^{n}, \ldots, u_{M-1}^{n}) \) explicitly defined by
\[
\begin{align*}
u_{m}^{n+1} &= u_{m}^{n} + \sum_{i=1}^{\nu} \left[ (A \varphi_{i} + B \omega_{i}) u_{m-1}^{(i)} - (2B \omega_{i}) u_{m}^{(i)} \\
&\quad + (B \omega_{i} - A \varphi_{i}) u_{m-1}^{(i)} \right], \quad \text{for } m = 0, \ldots, j - 1; \\
u_{m}^{n+1} &= u_{m}^{n} + \sum_{i=1}^{\nu} \left[ (A \varphi_{i} + B \omega_{i}) u_{m-1}^{(i)} - (2B \omega_{i} + \lambda \omega_{i}) u_{m}^{(i)} \\
&\quad + (B \omega_{i} - A \varphi_{i}) u_{m-1}^{(i)} + \lambda \varphi_{ij} \mathcal{I}_{h}(u_{m}^{(j)}) \right], \quad \text{for } m = j, \ldots, j + 1; \\
u_{m}^{n+1} &= u_{m}^{n} + \sum_{i=1}^{\nu} \left[ (A \varphi_{i} + B \omega_{i}) u_{m-1}^{(i)} - (2B \omega_{i}) u_{m}^{(i)} \\
&\quad + (B \omega_{i} - A \varphi_{i}) u_{m-1}^{(i)} \right], \quad \text{for } m = j + 1, \ldots, M - 1; \\
\end{align*}
\]
(7.2.6)

Both steps (S1)–(S2) need some additional conditions on the two boundary points \( x_{-1} \) and \( x_{M} \). We shall suppose that the solution has an asymptotic behavior such that
\[
u_{x}(x, \cdot) \approx f(x), \quad \text{as } x \to \pm \infty,
\]
for a given real function \( f \). For example, in the case of option pricing, \( f(x) \) may be chosen as the derivative of the payoff function. We then use the following second order approximation for first order derivatives,
\[
\frac{3u_{j} - 4u_{j+1} + u_{j+2}}{2h} = f(x_{j}),
\]
and we shall put this approximation on step (S1) to calculate the values \( u_{-1}^{(i)} \) and \( u_{M}^{(i)} \) for \( i = 1, \ldots, \nu \) and on step (S2) to calculate \( u_{n+1}^{(i)}, u_{M}^{n+1} \).

### 7.2.1 Computational costs

Let \( M \) be the number of space steps, \( N \) the number of time steps and \( \mathcal{P} = \text{card}(\mathcal{P}) \) the number of nodes for the integral approximation.

The direct integral approximation (7.1.3) gives at each space step a computational cost of order \( O(\mathcal{P}^2) \). This cost can decrease applying the discrete correlation theorem: the discrete Fourier transform of the discrete correlation of two real function \( g \) and \( h \) is such that

\[
< \text{Corr}(g, h)_j > = < g_i > < h_i^* > \tag{7.2.7}
\]

where \( < . > \) is the discrete Fourier transform operator, and the asterisk denotes complex conjugation. We can then compute correlation using the FFT as follows: FFT the two data sets, multiply one resulting transform by the complex conjugate of the other and inverse transform the product. The cost will be of \( O(3\mathcal{P} \log_2 (\mathcal{P}) + \mathcal{P}) \).

The cost to solve the tridiagonal system in point (S1) is \( O(M) \), as the cost for the computation of the \( M \) components \( \{ u_i^{n+1} \} \) given by (7.2.6). The global computational cost is then described in the table (7.1) and is of order \( O(\nu sN\mathcal{P} \log_2 \mathcal{P}) \). For second order schemes, because of the restriction \( k = O(h^{4/3}) \), one has \( N = O(M^{4/3}) \), and therefore the total computational cost for evolving the system up to a given time is \( O(sM^{7/3}\mathcal{P} \log \mathcal{P}) \). For third order schemes the stability restriction is of hyperbolic type, and therefore the complexity is \( O(M^2\mathcal{P} \log \mathcal{P}) \). The computational cost of an explicit time approximation of our problem is described in the table (7.2). Because in this case it is \( N = O(M^2) \), the overall complexity of the scheme is \( O(M^2\mathcal{P} \log \mathcal{P}) \).

Because the implicit step involves the solution of a tridiagonal system, the computational cost per time step in comparable to the cost of a fully explicit scheme.


\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{time integral computational cost} & \textbf{space} & \textbf{computational cost} \\
\hline
$N \times$ & $M \times P^2$ or $(3P \log_2 (P) + P)$ & $O(MN(3P \log_2 (P) + P))$ \\
& & $= O(M^3P \log_2 (P))$ \\
\hline
\end{tabular}
\end{table}

Table 7.2: computational cost for an explicit approximation of the PIDE (7.0.1)

\section{Numerical tests}

Let us consider the problem of pricing an European option according to the problem (7.0.1). As we have shown in Section 7.2, we solve the integro-differential equation (7.0.1) on the numerical domain $\bar{\Omega}$. We compute the order $\gamma$ of the error in the following form

$$
\gamma_{1,\infty} = \log_2 \left( \frac{e_{1,\infty}^h}{e_{1,\infty}^{h/2}} \right).
$$

The $l^1$ error $e_1^h$ is calculated as

$$
e_1^h = h \sum_j |v_j^h(T) - v_j^{h/2}(T)|,
$$

where $v_j^h(T)$ denotes the numerical solution obtained with the space step $h$ on node $x_j$ at time $T$, and we shall give the local $l^\infty$ error as

$$
e_{\infty}^h = |v^h(x^*, T) - v^{h/2}(x^*, T)|,
$$

where $x^*$ is the midpoint of the numerical domain.

Tables 7.3-7.4, 7.5-7.6, 7.7-7.8, show that the schemes (7.1.21), (7.1.22), (7.1.23) respectively, are locally second order in $l^\infty$. Tables 7.3-7.4, 7.5-7.6, 7.7-7.8, show also the global $l^1$ error (7.3.1). For all three schemes, the $\gamma_1$ order decreases, this is due to the boundary error. The boundary approximation presented in Section 7.2 is in fact a second order approximation under strong stability condition on the grid steps (see Proposition 6.5.2), a more accurate approximation is still an open field of investigation.

In Tables 7.9 and 7.10, we compare the price value at $S = 100$, obtained with an explicit approximation and the ARS-233 scheme (7.1.22) respectively. Comparing the price values, it is clear that we get the same order approximation. The advantage in using the IMEX ARS-233 scheme, as the Midpoint-122 or the SSP3-433 one, is the smaller number of time steps required to get the same result. This property influences of course also the computational time. Table 7.12 shows CPU times on a 1 GHz Pentium IV PC for various number of space steps. It is evident that the two IMEX approximation require less CPU time to give the solution.

Finally, in Table 7.11 we compute the put and call option prices at $S = 100$ using the Monte Carlo simulations, see Subsection 2.3.4. Notice that, to get a good
Implicit-Explicit Schemes

approximation we need a large number $n$ of iterations. Moreover, it is well known that Monte Carlo simulations are suitable to compute the price value at one point and not to compute the option price surface, as we show in Figure 7.8, where we compare the Midpoint-122 approximation to the solution of the integro-differential problem (7.0.1) with the Monte Carlo simulations.

7.4 Conclusions

In this paper we investigated the numerical solution of a European option pricing problem in a jump-diffusion market. We concentrated our investigation on the classical model by Merton [50].

Due to the non-local nature of the integro-differential operator, an implicit method is not appropriate, while the diffusion term if explicitly evaluated gives a strong restriction on the time step. The aim of this paper is to show how to avoid these two problems, applying an IMEX (Implicit-Explicit) approximation.

First, we have analyzed the behavior of an IMEX approximation, studying the order and the stability properties of these schemes. The IMEX schemes are high order schemes under weaker stability condition. We considered the pure differential case and we analyzed three approximation, a second order one, the Midpoint-122 rule and two third order schemes, called ARS-233 and SSP3-433. For these approximation we computed the CFL condition, showing that the Midpoint-122 is stable under the restriction $k = O(h^{4/3})$, while the ARS-233 and SSP3-433 are stable under the weak hyperbolic condition $k = O(h)$.

Then, we proved that adding the integral term to the problem does not change the quality of such approximation. If the integral term is bounded, the stability of the IMEX approximation applied to the pure differential problem guarantees the stability when considering the integro-differential equation.

Finally, we have shown how to implement such schemes, giving at the end some numerical results for the European option pricing problem. We compared the results with the solution given by a convergent explicit approximation; as we expected the IMEX schemes are globally accurate of the same order (due to the use of the same space approximation) but quite faster than the explicit ones.
Figure 7.1: Advection-diffusion problem: border of the stability regions in the $b\xi^2k-a\xi k$ plane.

Table 7.3: Errors and convergence orders of the Midpoint-122 scheme (7.1.21). The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$. 
Figure 7.2: Midpoint-122 (7.1.21): border of the stability region defined in (7.1.42) in the $h$-$k$ plane, varying the parameter $\lambda$. The bold line represents the CFL condition (7.1.35).

| $T = 5$ Midpoint-122 |
|---|---|---|---|---|---|
| h   | k     | $l^1$  | $\gamma_1$ | $l^\infty$ | $\gamma_\infty$ |
| 0.125000 | 0.066986 | 5.097443 | 0.606756  |
| 0.062500 | 0.027205 | 1.095825 | 2.217757  | 0.188375 | 1.687508 |
| 0.031250 | 0.011049 | 0.296188 | 1.887442  | 0.069759 | 1.433159 |
| 0.015625 | 0.004487 | 0.093908 | 1.657181  | 0.019274 | 1.855694 |
| 0.007812 | 0.001822 | 0.031876 | 1.558776  | 0.002183 | 3.142337 |

Table 7.4: Errors and convergence orders of the Midpoint-122 scheme (7.1.21). The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$. 
Figure 7.3: ARS-233 (7.1.22): border of the stability region defined in (7.1.42) in the $h$-$k$ plane, varying the parameter $\lambda$. The bold line represents the CFL condition (7.1.38).

<table>
<thead>
<tr>
<th>$T = 1$ ARS-233</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$k$</td>
<td>$l^1$</td>
<td>$\gamma_1$</td>
<td>$l^\infty$</td>
</tr>
<tr>
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<td>0.125000</td>
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<td>0.007812</td>
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<td>0.004429</td>
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</table>

Table 7.5: Errors and convergence orders of the ARS-233 scheme (7.1.22). The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$. 
Figure 7.4: SSP3-433 (7.1.23): border of the stability region defined in (7.1.42) in the $h$-$k$ plane, varying the parameter $\lambda$. The bold line represents the CFL condition (7.1.41).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$k$</th>
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<th>$\gamma_1$</th>
<th>$l^\infty$</th>
<th>$\gamma_\infty$</th>
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<tr>
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<td>0.031</td>
<td>0.007</td>
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</tr>
</tbody>
</table>

Table 7.6: Errors and convergence orders of the ARS-233 scheme (7.1.22). The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$. 
Figure 7.5: Integro-differential problem (6.5.1): Midpoint-122 (7.1.21) scheme, stability region \( R \) (gray domain) compared with the CFL condition (7.1.35) (−) and the hyperbolic condition \( k = h/a \) (⋯). The problem parameter are \( r = 0, \sigma = 0.2, \lambda = 0.1, \delta = 0.5 \).

\[
\begin{array}{cccccc}
\hline
T = 1 \text{ SSP3-433} & & & & & \\
0.125000 & 0.125000 & 1.345415 & & 1.409931 \\
0.062500 & 0.062500 & 0.338187 & 1.992159 & 0.319594 & 2.141312 \\
0.031250 & 0.031250 & 0.095421 & 1.825437 & 0.072456 & 2.141059 \\
0.015625 & 0.015625 & 0.030257 & 1.657046 & 0.017798 & 2.025392 \\
0.007812 & 0.007812 & 0.010545 & 1.520685 & 0.004431 & 2.005997 \\
\hline
\end{array}
\]

Table 7.7: Errors and convergence orders of the SSP3-433 scheme (7.1.23). The process parameters are \( E = 100, r = 0, \sigma = 0.2, \lambda = 0.1, \delta = 0.5 \).
Figure 7.6: Integro-differential problem (6.5.1): ARS-233 (7.1.22) scheme, stability region $R$ (gray domain) compared with the CFL condition (7.1.38) (-). The problem parameter are $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$k$</th>
<th>$l_1^1$</th>
<th>$\gamma_1$</th>
<th>$l_\infty^1$</th>
<th>$\gamma_\infty$</th>
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</thead>
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<td>2.000878</td>
</tr>
</tbody>
</table>

Table 7.8: Errors and convergence orders of the SSP3-433 scheme (7.1.23). The process parameters are $E = 100$, $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$. 
Figure 7.7: Integro-differential problem (6.5.1): SSP3-433 (7.1.23) scheme, stability region $R$ (gray domain) compared with the CFL condition (7.1.41) (-). The problem parameter are $r = 0$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.5$.

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>T=1</th>
<th></th>
</tr>
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<tbody>
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<td></td>
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<td>185</td>
<td>8.268306</td>
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<td>1024</td>
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<tr>
<td>V</td>
<td></td>
<td>8.341444</td>
<td>13.218501</td>
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</table>

Table 7.9: $V$ is the analytical price of the European put and call option of Merton model. The process parameters are $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.8$, $x = \ln(100)$. 
Table 7.10: $V$ is the analytical price of the European put and call option of Merton model (2.3.8). The process parameters are $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.8$, $x = \ln(100)$.

Table 7.11: $V$ is the analytical price of the European put and call option of Merton model (2.3.8). The process parameters are $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.8$, $x = \ln(100)$ and $n$ is the number of iterations for the Monte Carlo Simulations, see Chapter 2.

Table 7.12: CPU times on 1.6 GHz Pentium IV PC when $T = 1$. 
Figure 7.8: We show the call option price (−) computed by the Midpoint-122 approximation to the solution of the integro-differential equation (7.0.1), compared with the call option price (×) computed by the Monte Carlo algorithm given in Subsection 2.3.4
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