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The Idempotent Analog of Resolvent Kernels for a Deterministic Optimal Control Problem

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Abstract—A solution of a discrete Hamilton–Jacobi–Bellman equation is represented in terms of idempotent analysis as a convergent series of integral operators.

Key words: *idempotent analysis, optimal control problem, Hamilton–Jacobi–Bellman equation, Volterra equation, correspondence principle, resolvent kernels.*

INTRODUCTION.

In many cases, the use of the semiring $(\mathbb{R}, \min, +)$ turns out to be fundamental in order to apply constructions developed for linear operators to nonlinear ones. It is well known that the discrete Bellman equation can be treated as linear over appropriate idempotent semirings, and in the paper, one of the possible directions arising from idempotent analysis is explored: namely, the discretized Hamilton–Jacobi–Bellman equation turns out to be an integral equation and its study can contribute to the development of functional analysis methods such as the ones used in the approximation of the Volterra equation by means of resolvent kernels.

Of course, our method is neither the most general in idempotent analysis nor are the conditions which permit to point out this connection widely assumed in the theory of optimal control, but they allow us to show that it is also possible to apply the analogy between idempotent analysis and functional analysis in a wider way, finding useful suggestions in the way of treating approximations. The main result obtained is the representation formula for the value function in terms of linear operators. This permits to show important properties and to consider the issue of using parallel algorithms developed for linear algebra in nonlinear cases. In the framework of idempotent analysis, the explicit construction of the solution is simplified. We give a meaningful analog to the theorems and propositions we prove in the spirit of the correspondence principle. Instead of remaining in the context of heuristic considerations, we explore several aspects in detail. In [1] the correspondence principle is used to develop an approach to object-oriented software and hardware design for the algorithms of idempotent calculus and scientific calculations. Our aim is to particularize these general and heuristic aspects of idempotent calculus to our case and to study certain details. This permits us to present some considerations about a possible strategy. First of all, carrying over the problem to symbolic calculus can be considered, including the idea of using universal algorithms. On the other hand, many other efforts are necessary to formulate general computing strategies that take into account the new linearity of the problem. This will be the object of a forthcoming publication.

Let us briefly describe the optimal control problem we are interested in. We consider a dynamic process where we can act by choosing control functions and stopping times. We assume that two functions are considered: the running cost f and the stopping cost ψ, depending on the state process and on the controls. We must minimize the given functional over all the controls α and over all the stopping times θ, i.e., we want to determine the value function:

$$u(x) = \inf_{\alpha,\theta} J(x, \alpha, \theta), \qquad x \in \mathbb{R}^n.$$

It is well known (we refer for instance to [2]) that this problem can be seen as a deterministic infinite horizon problem in which the set of controls is allowed to be larger (containing zero in the set of values), and correspondingly an extra component is added to the velocity field and to the cost function.

On the other hand, the converse is true only if there exists a control function such that $b(X(t), \alpha(t)) = 0$ for every t.

The Hamilton–Jacobi equation associated with the stopping time problem is a variational inequality of obstacle type.

Let us briefly describe the contents of this paper. In Sec. 1 we recall the optimal control problem we are interested in and its discretization. In Sec. 2 we rewrite the problem using an appropriate algebraic structure for linearizing it and we find the analog of the Dirichlet formula for the idempotent calculus. Finally, in Sec. 3 we state and prove the main results, based on the integral representation of the solution.

1. THE OPTIMAL CONTROL PROBLEM

We consider the optimal deterministic control problem of stopping time. We assume that the state of the system is given by the solution of the ordinary differential equation

$$\dot{X}(t) = b(X(t), \alpha(t)), \qquad X(0) = x \in \mathbb{R}^n. \tag{1}$$

In (1) b is a continuous function from $\mathbb{R}^n \times A$ to $\mathbb{R}^n$, where A (a compact subset of $\mathbb{R}^m$) will be called the set of the values of the control. Moreover, to ensure existence and uniqueness of the solution (in $\mathrm{Lip}[0, T]$ for any $T > 0$) we assume that for all $(x, x', a) \in \mathbb{R}^n \times \mathbb{R}^n \times A$, there exist numbers $M_b, L_b > 0$ such that

$$|b(x, a) - b(x', a)| \le L_b|x - x'|, \qquad |b(x, a)| \le M_b.$$

Next we consider the cost functions given by

$$J(x, \alpha(t), \theta) = \int_0^\theta f\bigl(X(t), \alpha(t)\bigr)e^{-\lambda t}\,dt + \psi(X(\theta))e^{-\lambda\theta},$$

where $\psi \in BUC(\mathbb{R}^n)$ (bounded uniformly continuous in $\mathbb{R}^n$) and λ is a positive parameter. We assume that for all $(x, x', a) \in \mathbb{R}^n \times \mathbb{R}^n \times A$ there exist $L_f, M_f > 0$ such that

$$|f(x, a) - f(x', a)| \le L_f|x - x'|, \qquad |f(x, a)| \le M_f.$$

Here the control is given by the couple $\alpha(\,\cdot\,)$ and the stopping time θ. This means that the process can be stopped at any time, paying the stopping cost ψ. The minimum cost function (or the value function) is defined as

$$u(x) = \inf_{\alpha,\theta} J(x, \alpha, \theta) \qquad x \in \mathbb{R}^n, \quad \theta \ge 0.$$

The Hamilton-Jacobi-Bellman equation

$$\max\Bigl\{u(x) - \psi(x),\, \lambda u(x) + \max_{a\in A}\{-b(x, a)Du(x) - f(x, a)\}\Bigr\} = 0$$

is the optimality condition for this deterministic optimal control problem. Indeed, assuming C^1-regularity for the value function $u(x)$ and the continuity property for the control, it is possible to determine the optimal feedback control. The cases of loss of regularity for the value are so easy to construct that the arguments above are rarely applied. The development of these topics lead to the use of the notion of viscosity solution, introduced by Crandall and Lions in a more general context [3].

As in [4], the discretized equation is given by

$$u_h(x) = \min\Big\{\min_{a\in A}\{(1-\lambda h)u_h(x+hb(x,a)) + hf(x,a)\},\ \psi(x)\Big\}. \tag{2}$$

2. THE IDEMPOTENT ANALOG

In this section we recall the traditional material used by Maslov et al. in the interpretation of differential equations in idempotent semirings: in order to match equation (2) with the pattern of the Volterra integral equation of 2nd type, a change of variable will be introduced.

The semiring $\mathbb{R}_{\min}$ that our problem suggests to consider is given by the set $\mathbb{R}\cup\{+\infty\}$ endowed with the two operations $a\oplus b\equiv\min(a,b)$ and $a\odot b\equiv a+b$, the identity element $\mathbf{0}$ with respect to $\oplus$ is $+\infty$ and with respect to $\odot$ is $\mathbf{1}\equiv 0$. This semiring is supplied with a "measure" defined by

$$\int_X^{\oplus} f(x)\odot g(x)\,dx = \inf_{x\in X}\{f(x)+g(x)\}.$$

Just as ordinary integrals, idempotent integrals possess the following properties:

$$\int_X^{\oplus} f(x)\,dx \oplus \int_X^{\oplus} g(x)\,dx = \int_X^{\oplus} f(x)\oplus g(x)\,dx\,;$$

indeed,

$$\min\Big\{\inf_{x\in X} f(x),\ \inf_{x\in X} g(x)\Big\} = \inf_{x\in X}\min\big\{f(x),g(x)\big\}\,;$$

moreover, for every $k\in\mathbb{R}_{\min}$ we have

$$\int_X^{\oplus} k\odot f(x)\,dx = k\odot\int_X^{\oplus} f(x)\,dx\,;$$

indeed,

$$\inf_{x\in X}\{k+f(x)\} = k+\inf_{x\in X} f(x)\,;$$

further,

$$\int_{X\cup Y}^{\oplus} f(x)\,dx = \int_X^{\oplus} f(x)\,dx \oplus \int_Y^{\oplus} f(x)\,dx\,;$$

indeed,

$$\inf_{x\in X\cup Y} f(x) = \min\Big\{\inf_{x\in X} f(x),\ \inf_{x\in Y} f(x)\Big\}.$$

If the additive operation of the semiring is idempotent, it is possible to introduce a partial order in the following way: $a\preccurlyeq b$ if and only if $a\oplus b=b$; in this case several algebraic properties can be derived:

(i) $a\preccurlyeq b,\mathbf{1}\preccurlyeq c \implies a\preccurlyeq b\odot c$,

(ii) $\forall c, a\preccurlyeq b \implies c\odot a\preccurlyeq c\odot b$,

(iii) $a\preccurlyeq b, \mathbf{0}\preccurlyeq c\preccurlyeq\mathbf{1} \implies c\odot a\preccurlyeq b$,

(iv) $f\preccurlyeq g \implies \int_X^{\oplus} f(x)\,dx \preccurlyeq \int_X^{\oplus} g(x)\,dx$

Another important result involving the definition of measure satisfied by idempotent semirings is the analog of the Fubini theorem, which states that it is possible to reverse the order of integration in multiple integrals:

$$\int_X^{\oplus} \int_Y^{\oplus} \varphi(x, y)\, dy\, dx = \int_Y^{\oplus} \int_X^{\oplus} \varphi(x, y)\, dx\, dy.$$

In the particular semiring that we are considering, this is easy to prove:

$$\inf_{x \in X} \inf_{y \in Y} \varphi(x, y) = \inf_{y \in Y} \inf_{x \in X} \varphi(x, y).$$

In the sequel, a more general form of this relation will be used in order to reverse the order of integrals summed over sets depending on the integration variable of the outer integral. The following formula is a consequence of the Fubini theorem and is valid in idempotent semirings as well as in more traditional settings (indeed, it is used also in the original proof of the method of resolvent kernels in [5]):

$$\int_X dx \int_{F(x)} \varphi(x, y)\, dy = \int_{F(X)} dy \int_{F^{-1}(y)} \varphi(x, y)\, dx.$$

Let us specify the notation: we suppose that F is a function associating a subset of Y to an element $x \in X$, (i.e., $F(x) \subset Y$) and we write $F^{-1}(y) = \{x \colon y \in F(x)\}$.

As the reader will observe, besides idempotent operations, ordinary multiplication can occur in the expressions that we consider, and in the semiring $\mathbb{R}_{\min}$ the following laws can be applied: for every $\lambda \in \mathbb{R}_{\min}$ we have

$$\lambda \mathbf{1} = \mathbf{1}$$

and whenever $\lambda \geq 0$, we have

$$\lambda \biggl(\int_X^{\oplus} f(x) \odot g(x)\, dx \biggr) = \int_X^{\oplus} (\lambda f(x)) \odot (\lambda g(x))\, dx.$$

To conclude this section, we give the idempotent rewriting of the discretized Hamilton-Jacobi-Bellmann equation (2):

$$u_h(x) = \psi(x) \oplus \tilde{\lambda} \int_{B(x,A)}^{\oplus} u_h(\xi_0) \odot G_h(x, \xi_0)\, d\xi_0, \tag{3}$$

where $\tilde{\lambda} = (1 - \lambda h)$, $B(x, a) = x + hb(x, a)$ and

$$G_h(x, \xi) = \frac{h}{1 - \lambda h} \min \biggl\{ f(x, a);\ a \in \pi_2 \biggl(b^{-1} \biggl(\frac{\xi - x}{h} \biggr) \biggr) \biggr\}.$$

Here π_2 is the projection operator $\mathbb{R}^n \times A \to A$.

3. RESOLVENT KERNELS AND DYNAMIC PROGRAMMING

The integral equation (3), whenever expressed in terms of ordinary integrals, can be treated by different methods. A first approach studied by Maslov for the homogeneous case consists in the analog of spectral theory in idempotent semirings. In this section we will follow a different outline: the solution obtained in [4, 6] by successive approximations for the original equation will be read in its idempotent version as the resolvent kernel associated to the integral equation.

The function $G_h(x,\xi_0)$ is called the *kernel* of the equation, so its solution is expressed by the formula

$$u_h(x)=\psi(x)\oplus\bigoplus_{n=1}^{\infty}\tilde\lambda^n\int_{B^n(x,A)}^{\oplus}\psi(t)\odot K^{(n)}(t,x)\,dt, \tag{4}$$

where the *iterated kernels* $K^{(n)}(t,x)$ are defined by

$$\begin{cases} K^{(1)}(t,x)=G_h(x,t), & t\in B^{(p)}(x,A), \qquad p\ge 1,\\ K^{(n+1)}(t,x)=\displaystyle\int_{B^{-1}(t)}^{\oplus}K^{(1)}(t,\xi)\odot\frac{K^{(n)}(\xi,x)}{\tilde\lambda}\,d\xi, & t\in B^{(p)}(x,A), \qquad p\ge n+1.\end{cases}$$

In the following, we will show that formula (4) actually gives a solution for (3) and that the solution is unique. It is assumed that h is small enough (see the remark after Theorem 3.3 below).

In order to obtain (4), we treat the equation by the method of successive approximations: in equation (3), all the occurrences of the value function $u_h(x)$ are replaced by the right-hand side of (3) itself, and the following relation will be proved by induction over p.

Proposition 3.1. *For every integer p, we have*

$$u_h(x)=\psi(x)\oplus\bigoplus_{n=1}^{p}\tilde\lambda^n\int_{B^n(x,A)}^{\oplus}\psi(t)\odot K^{(n)}(t,x)\,dt \oplus\tilde\lambda^{p+1}\int_{B^{p+1}(x,A)}^{\oplus}u_h(t)\odot K^{(p+1)}(t,x)\,dt, \tag{5}$$

where $B^n(x,A)=\{B(B(\dots(B(x,\alpha_1),\alpha_2),\dots),\alpha_n)\mid \alpha_i\in A\}$.

Proof. For $p=1$ we have already proved this. The relation

$$u_h(x)=\psi(x)\oplus\tilde\lambda\int_{B(x,A)}^{\oplus}\psi(t)\odot K^{(1)}(t,x)\,dt\oplus\tilde\lambda^2\int_{B^2(x,A)}^{\oplus}u_h(t)\odot K^{(2)}(t,x)\,dt$$

is obtained by substituting $u_h(x)$ in (3), where

$$u_h(x)=\psi(x)\oplus\tilde\lambda\int_{B(x,A)}^{\oplus}\left[\psi(\xi_0)\oplus\tilde\lambda\int_{B(\xi_0,A)}^{\oplus}u_h(\xi_1)\odot G_h(\xi_0,\xi_1)\,d\xi_1\right]\odot G_h(x,\xi_0)\,d\xi_0.$$

Since $\tilde\lambda>0$, by distributivity we have

$$\begin{aligned} u_h(x)&=\psi(x)\oplus\tilde\lambda\int_{B(x,A)}^{\oplus}\psi(\xi_0)\odot K^{(1)}(\xi_0,x)\,d\xi_0\\ &\quad\oplus\tilde\lambda\int_{B(x,A)}^{\oplus}\left[\tilde\lambda\int_{B(\xi_0,A)}^{\oplus}u_h(\xi_1)\odot K^{(1)}(\xi_1,\xi_0)\,d\xi_1\right]\odot K^{(1)}(\xi_0,x)\,d\xi_0\\ &=\psi(x)\oplus\tilde\lambda\int_{B(x,A)}^{\oplus}\psi(\xi_0)\odot K^{(1)}(\xi_0,x)\,d\xi_0\\ &\quad\oplus\tilde\lambda^2\int_{B(x,A)}^{\oplus}\left[\int_{B(\xi_0,A)}^{\oplus}u_h(\xi_1)\odot K^{(1)}(\xi_1,\xi_0)\,d\xi_1\right]\odot\frac{K^{(1)}(\xi_0,x)}{\tilde\lambda}\,d\xi_0.\end{aligned}$$

Now by the Dirichlet formula

$$\begin{aligned} u_h(x)&=\psi(x)\oplus\tilde\lambda\int_{B(x,A)}^{\oplus}\psi(\xi_0)\odot K^{(1)}(\xi_0,x)\,d\xi_0\\ &\quad\oplus\tilde\lambda^2\int_{B(B(x,A),A)}^{\oplus}u_h(\xi_1)\odot\int_{B^{-1}(\xi_1)}^{\oplus}K^{(1)}(\xi_1,\xi_0)\odot\frac{K^{(1)}(\xi_0,x)}{\tilde\lambda}\,d\xi_0\,d\xi_1.\end{aligned}$$

By the induction hypothesis formula (5) is valid, so we can apply another step of the approximation process (i.e., by substituting $u_h(x)$ into the last term of the right-hand side of (3)):

$$u_h(x) = \psi(x) \oplus \bigoplus_{n=1}^{p} \tilde{\lambda}^n \int^{\oplus}_{B^n(x,A)} \psi(t) \odot K^{(n)}(t,x)\,dt \\ \oplus \tilde{\lambda}^{p+1} \int^{\oplus}_{B^{p+1}(x,A)} \left[\psi(t) \oplus \tilde{\lambda} \int^{\oplus}_{B(x,A)} u_h(\xi_p) \odot G_h(x,\xi_p)\,d\xi_p\right] \odot K^{(p+1)}(t,x)\,dt,$$

so by distributivity we obtain

$$u_h(x) = \psi(x) \oplus \bigoplus_{n=1}^{p+1} \tilde{\lambda}^n \int^{\oplus}_{B^n(x,A)} \psi(t) \odot K^{(n)}(t,x)\,dt \\ \oplus \tilde{\lambda}^{p+1} \int^{\oplus}_{B^{p+1}(x,A)} \left[\tilde{\lambda} \int^{\oplus}_{B(x,A)} u_h(\xi_p) \odot G_h(x,\xi_p)\,d\xi_p\right] \odot K^{(p+1)}(t,x)\,dt.$$

By the Dirichlet formula

$$u_h(x) = \psi(x) \oplus \bigoplus_{n=1}^{p+1} \tilde{\lambda}^n \int^{\oplus}_{B^n(x,A)} \psi(t) \odot K^{(n)}(t,x)\,dt \\ \oplus \tilde{\lambda}^{p+2} \int^{\oplus}_{B(B^{p+1}(x,A),A)} u_h(\xi_p) \odot \left[\int^{\oplus}_{B^{(-1)}(\xi_p)} \odot K^{(1)}(\xi_p,x) \odot K^{(p+1)}(t,x)\,dt\right] d\xi_p.$$

Now, by the definition of the iterated kernel, we can write

$$u_h(x) = \psi(x) \oplus \bigoplus_{n=1}^{p+1} \tilde{\lambda}^n \int^{\oplus}_{B^n(x,A)} \psi(t) \odot K^{(n)}(t,x)\,dt \\ \oplus \tilde{\lambda}^{p+2} \int^{\oplus}_{B^{p+2}(x,A)} u_h(\xi_p) \odot K^{(p+2)}(\xi_p,x)\,d\xi_p. \quad \square$$

It remains to show that the value function is well defined. Formula (4) makes sense if the functions appearing in the sum,

$$A_n = \tilde{\lambda}^n \int^{\oplus}_{B^n(x,A)} \psi(t) \odot K^{(n)}(t,x)\,dt,$$

can be shown to be bounded, in which case the convergence of the idempotent series will be proved. That means that there exists an integer n such that the least upper bound exists.

The following property shows the convergence of the series.

Proposition 3.2. *For every integer n, the function $\lambda^n|K^{(n)}(t,x)|$ is bounded.*

Proof. First of all we show that the iterated kernels are bounded:

$$M_f \preccurlyeq \left|\min_{a\in\pi_2(b^{-1}(\xi-x)/h)} f(x,a)\right|.$$

Recall that π_2 is the projection operator $\mathbb{R}^n \times A \to A$. Taking $m_h = hM_f$, we obtain

$$|G_h(x,\xi)| \preccurlyeq m_h/\tilde{\lambda},$$

and by definition

$$m_h \preccurlyeq \tilde{\lambda}|K^{(1)}(t,x)|.$$

By induction suppose that

$$m_h \bigodot_{k=0}^{n-1} \tilde{\lambda}^k \preccurlyeq \tilde{\lambda}^n |K^{(n)}(t,x)|\,;$$

let us show that

$$m_h \bigodot_{k=0}^{n} \tilde{\lambda}^k \preccurlyeq \tilde{\lambda}^{n+1} |K^{(n+1)}(t,x)|.$$

By definition

$$\begin{aligned}\tilde{\lambda}^{n+1} K^{(n+1)}(t,x) &= \int_{B^{-1}(t)}^{\oplus} \tilde{\lambda}^{n+1} K^{(1)}(t,\xi) \odot \frac{\tilde{\lambda}^{n+1} K^{(n)}(\xi,x)}{\tilde{\lambda}}\, d\xi \\ &= \int_{B^{-1}(t)}^{\oplus} \tilde{\lambda}^{n+1} K^{(1)}(t,\xi) \odot \tilde{\lambda}^{n} K^{(n)}(\xi,x)\, d\xi,\end{aligned}$$

and so

$$m_h \tilde{\lambda}^n \odot m_h \bigodot_{k=0}^{n-1} \tilde{\lambda}^k \preccurlyeq |\tilde{\lambda}^{n+1} K^{(n+1)}(t,x)|.$$

Since $m_h \bigodot_{k=0}^{\infty} \tilde{\lambda}^k$ is convergent, say to $m_K = m_h/(1-\tilde{\lambda})$, we know that the generic element A_n of the idempotent series is bounded, in fact,

$$m_\psi \odot m_K \preccurlyeq \tilde{\lambda}^n \left| \int_{B^n(x,A)}^{\oplus} \psi(t) \odot K^{(n)}(t,x)\, dt \right|,$$

where $m_\psi \preccurlyeq |\psi(x)|$, and so the idempotent series $\bigoplus_{n=1}^{\infty} A_n$ is convergent too. □

In order to express the uniqueness of the solution, we recall a well-known argument. Let

$$T' u_h(x) = \tilde{\lambda} \int_{B(x,A)}^{\oplus} u_h(t) \odot G_h(x,t)\, dt$$

and $T u_h(x) = \psi(x) \oplus T' u_h(x)$. We will establish that T is a contraction. Indeed, we have $|T u_h^1 - T u_h^2| \le |T' u_h^1 - T' u_h^2|$. Now suppose that $\hat{t}$ satisfies $T' u_h^2(x) = \tilde{\lambda}(u_h^2(\hat{t}\,) \odot G(x,\hat{t}\,))$, so that

$$|T' u_h^1(x) - T' u_h^2(x)| \le |\tilde{\lambda}(u_h^1(\hat{t}\,) \odot G_h(x,\hat{t}\,) - u_h^2(\hat{t}) \odot G_h(x,\hat{t}))| = \tilde{\lambda}|u_h^1(\hat{t}\,) - u_h^2(\hat{t}\,)|,$$

i.e., we have

$$\|T u_h^1 - T u_h^2\|_\infty \le \tilde{\lambda} \|u_h^1 - u_h^2\|_\infty,$$

and this implies the uniqueness of the function u_h satisfying $T u_h = u_h$.

The *discrete dynamic programming principle* applies in the p-step approximation of the homogeneous Volterra equation for h small enough.

Theorem 3.3. *For every $x \in \mathbb{R}^n$ such that $u_h(x) < \psi(x)$, there exists an integer p_0 such that for every $p \le p_0$:*

$$u_h(x) = \tilde{\lambda}^p \int_{B^p(x,A)}^{\oplus} u_h(t) \odot K^{(p)}(t,x)\, dt.$$

Proof. By the continuity of $u_h(x)$ and $\psi(x)$, there exists a ball N_x centered at x such that for every $z \in N_x \cap B^p(x, A)$ the condition $u_h(z) < \psi(z)$ holds. Thus equation (3) can be considered as being homogeneous, and by iterated approximation, the theorem can be obtained by following the proof of Proposition 3.1. □

Remark. In order to apply the statement of the theorem, h must be small enough in the sense that there exists an h_0 such that for every $h \le h_0$, if $u_h(x) < \psi(x)$, then $u_h(z) < \psi(z)$ for every $z = B_h(x, A)$.

So, considering the approximate solution $u_h(x)$ of equation (3), the feedback law F can be explicitly defined:

Theorem 3.4. *Suppose we are given a map $F\colon \mathbb{R}^n \to A$ such that for every x and for $z = B_h(x, F(x))$ there exists an h_0 such that for every $h < h_0$ if $u_h(x) < \psi(x)$, then $u_h(z) < \psi(z)$ and $u_h(x) = \tilde\lambda[u_h(z) \odot G_h(x, z)]$. If the sequence $(z_p(x))_{p\in\mathbb{N}}$ is defined by*

$$\begin{cases} z_0(x) = x, \\ z_{p+1}(x) = B(z_p(x), F(z_p(x))), \end{cases}$$

and the iterated kernels is defined by

$$\begin{cases} \overline{K}_1(x) = G_h(x, z_1(x)), \\ \overline{K}_{p+1}(x) = G_h(z_p(x), z_{p+1}(x)) \odot \dfrac{\overline{K}_p(x)}{\tilde\lambda}, \end{cases}$$

then $(z_p(x))_{p\in\mathbb{N}}$ describes an optimal trajectory, i.e.,

$$u_h(x) = \tilde\lambda^{p+1}[u_h(z_{p+1}(x)) \odot \overline{K}_p(x)].$$

Proof. The Hamilton–Jacobi–Bellman equation (3) is of the form

$$u_h(x) = \psi(x) \oplus \tilde\lambda \int_{B(x,A)}^{\oplus} u_h(\xi_0) \odot G_h(x, \xi_0)\, d\xi_0.$$

Indeed, for $u_h(x) < \psi(x)$ we have

$$u_h(x) = \tilde\lambda \int_{B(x,A)}^{\oplus} u_h(\xi_0) \odot G_h(x, \xi_0)\, d\xi_0,$$

and since $u_h(\cdot)$ and $G_h(\cdot)$ are bounded functions, the integral has a mean value z_x, i.e.:

$$u_h(x) = \tilde\lambda[u_h(z_x) \odot G_h(x, z_x)].$$

So $F(x) = a_x$, where a_x satisfies $z_x = B(x, a_x)$.

Moreover, as a corollary of the discrete dynamic programming principle, we conclude that the sequence $(z_p(x))_{p\in\mathbb{N}}$ is the optimal trajectory. □

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