GLOBAL EXISTENCE OF SOLUTIONS TO A NONLINEAR MODEL OF SULPHATION PHENOMENA IN CALCIUM CARBONATE STONES

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Abstract. We prove global existence and uniqueness of smooth solutions to a nonlinear system of parabolic equations, which arises to describe the evolution of the chemical aggression due to the action of sulphur dioxide on calcium carbonate stones. This system is not strongly parabolic and only some energy estimates are available. Nevertheless, global (in time) results are proven using a weak continuation principle for the local solutions.

1. Introduction

In this paper we investigate the reaction diffusion system

$$\left\{ \begin{aligned} \partial_t(\varphi(c)s) &= \partial_x(\varphi(c)\partial_x s) - \varphi(c)cs, \\ \partial_t c &= -\varphi(c)cs, \end{aligned} \right. \tag{1.1}$$

for $(x,t) \in \mathbb{R} \times [0,T]$ $(T > 0)$, under the initial conditions

$$s(x,0) = s_0(x),\ c(x,0) = c_0(x). \tag{1.2}$$

This model has been introduced in [2] to describe the transformation in time of $CaCO_3$ (calcium carbonate) stones under the chemical aggression due to the SO_2 (sulphur dioxide). In (1.1), s stands for the porous concentration of SO_2, namely the concentration taken with respect to the volume of the pores, and c for the local density of $CaCO_3$.

In the following, we always assume that the initial data $s_0,\ c_0$ are nonnegative functions such that

$$0 \le s_0(x) \le S_0\ ,\ 0 < m \le c_0(x) \le C_0\ ,\ s_0 \in L^2(\mathbb{R})\ ,\ C_0 - c_0 \in H^1(\mathbb{R}). \tag{1.3}$$

The function φ (the porosity) is a linear function of the density c, $\varphi(c) = A + Bc$, which is strictly positive on the interval $[0, C_0]$; actually there are two strictly positive constants $\varphi_m < \varphi_M$, such that

$$\varphi_m \le \varphi(c) \le \varphi_M, \qquad \text{for all } c \in [0, C_0].$$

In particular, this implies that $\min\{A, A + BC_0\} \ge \varphi_m > 0$.

There is an extensive chemical literature about the deterioration mechanisms of natural building stones [14, 15, 10, 7, 9], which deals with problems concerning modern and historical buildings. Sulphur dioxide and nitrogen oxides emitted into the atmosphere by sources related to industry, transportation and heating, react

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with calcium carbonate stones to form sulphates and nitrates, which, due to their solubility in water, may be drained away or, if protected from the rain, may form crusts, that eventually exfoliate, see [7]. Standard methods developed for studying the evolution of this kind of damage use the statistic determination on the ratio dose/response of the materials.

The model (1.1), proposed in [2], is based on the different framework of hydrodynamical models, and gives the possibility of an accurate numerical approximation of the equations by finite elements or finite differences methods. Other advantages of this formulation are: a better understanding of the involved physical processes, the possibility to adapt the model to more complex situations, where other damage factors are involved, and an affective calibration against experimental data, see [6] for some preliminary results. One main factor is the possibility of a determination of the time asymptotic regime in one space dimension, which has been numerically explored in [2].

In this paper we are interested in the proof of existence, uniqueness and regularity of global solutions to problem (1.1)–(1.2). In our investigation we have to take into account that we are dealing with a system of nonlinear parabolic equations, which is not parabolic in the sense of Petrovskii [18]. Nevertheless, existence and uniqueness of local solutions are known, at least for smooth initial data, see for instance [5].

To obtain global solutions, we have to use some kind of continuation principle. Standard criteria, as the uniform boundedness of some Hölder-norm of the solutions [18], or the dependence of the life-span only on the L^∞-norm of the data [16], do not work in our case. Besides, system (1.1) does not verify the coupling conditions of Kawashima-Shizuta [13] for the global existence of solutions to general systems of hyperbolic-parabolic type.

The main difficulty concerns the uniform global estimates of the spatial derivative of c. Actually, the numerical tests performed in [2], show that the solution approaches, for large times, a free boundary problem with a discontinuous limit for the calcium carbonate density. Therefore the function c_x is expected to blow-up as $t \to +\infty$ in all the L^p-norms, for $1 \le p \le \infty$.

Observe that the same problem, but with constant porosity $\varphi(c) = \overline{\varphi}$, has been studied in [8]. In [17], similar techniques were used to investigate a model with variable porosity, but constant diffusion. In both papers, the authors integrate the ordinary differential equation to obtain c in function of $z(x,t) = \displaystyle\int_0^t s(x,\tau)d\tau$. Therefore, thanks to the assumption of constant diffusivity, they can show that the system is equivalent to a scalar heat equation with source for the new unknown z. In this way, they obtain the global existence result for the original system, and actually a rigorous analysis of the time asymptotic behavior of the solutions.

Clearly also in the present case it is possible to consider the system as one parabolic equation coupled with an ordinary differential equation. However we have to take into account the strong coupling between s and c into the divergence form, which makes impossible to reduce our problem to a scalar parabolic one. In fact, the integration of the ordinary differential equation for c yields

$$c(x,t) = \frac{Ac_0(x)}{\varphi(c_0)e^{A\int_0^t s(x,\tau)d\tau} - Bc_0(x)}. \tag{1.4}$$

Now we can write the parabolic equation for s as

$$s_t = s_{xx} + \frac{Bc_x}{\varphi(c)} s_x + cs(Bs-1)\ ,$$

where c_x is obtained by deriving (1.4). The above equation is nonlocal, due to the dependence of c on s. Moreover, because of the lack of estimates on the quantity

c_x, here considered as a coefficient, the assumptions of classical existence theorems for nonlinear scalar parabolic equations are not verified.

The main strategy in the present paper is to prove a **new** local existence result for suitable weak solutions, with a refined continuation principle, based on the **time local** control of c_x in L^2. It turns out that we can prove this weaker bound for our model and then the global continuation of the solutions.

A similar approach has been independently proposed in [4], to deal with weak solutions for a system which arises in modeling angiogenesis, and coincides with system (1.1), apart from the terms of order zero in the first equation. However, in that note the regularity problem was not addressed.

The paper is organized as follows. In the next section we shortly describe our model, starting from the main ideas of macroscopic modeling of filtration in porous media [3, 12]. In Section 3, we introduce a linearized version of our problem and we prove some estimates for its solutions. In the last section, using the results of Section 3, existence, uniqueness and regularity of solutions to problem (1.1)–(1.2) are proven.

2. Physical motivations

The mathematical description of the time evolution of sulphation process has been first proposed in [10], to give some measurements of the main physico–chemical parameters. Different regions and time regimes were described by different parameters and models, and then matched to fit the experimental behavior of the reaction. Here we consider a single mathematical model developed in [2], which in principle is able to take into account for the full behavior of the solutions. The model is derived by using basic physical relations, as the balance laws of the chemical reactions and the Fick's law, and by neglecting the permeability of the medium.

The path of reaction of SO_2 with calcite is assumed to be governed by a simplified one-step reaction,

$$CaCO_3 + SO_2 + \frac{1}{2}O_2 + 2H_2O \to CaSO_4 \cdot 2H_2O + CO_2. \tag{2.1}$$

This means that for one molecules of calcium carbonate and one of sulphur dioxide, adding oxygen and water, the reaction produces one molecule of calcium sulphate (gypsum) and carbon dioxide. Here, we neglect all heat effects, and assume the air contains enough water to give rise to the reaction. Moreover, we assume that changes in concentration of oxygen, water, and carbon dioxide do not affect the reaction.

Let Ω be the domain occupied by the specimen of calcite under consideration and set ρ_s for the total concentration of SO_2, c for the density of calcite and γ for the density of gypsum. Following [1, 3, 10], we assume that ρ_s and c satisfy the following balance laws

$$\begin{cases} \partial_t \rho_s + \nabla \cdot (\rho_s \mathbf{V}_s) &= -\frac{k}{m_c}\rho_s c, \\ \\ \partial_t c &= -\frac{k}{m_s}\rho_s c, \end{cases} \tag{2.2}$$

where $\mathbf{V}_s$ is the sulphur dioxide "fluid" velocity, k is the (constant) reaction rate, m_c, m_s are the masses of single molecules of calcite and SO_2.

Notice that we assume the conservation of the total density, and then we can recover the density of gypsum as a function of calcite by the formula

$$c + \frac{m_c}{m_\gamma}\gamma = c_0 + \frac{m_c}{m_\gamma}\gamma_0, \tag{2.3}$$

for some given initial densities c_0 and γ_0, respectively for calcite and gypsum, and setting m_γ for the mass of a molecule of gypsum.

Next, we introduce the porosity of the calcite specimen φ, which is not assumed to be constant, since the transformation of calcite in gypsum alters the volume of void (occupied by air and sulphur dioxide). Therefore, it is reasonable to regard it as a function of the amount of gypsum or, equivalently, as a function of the amount of calcite, that is $\varphi = \varphi(c)$. As shown in [1], the porosity of the specimen during the reaction can be expressed as a linear combination of the porosity of the pure calcite specimen, φ_0, and the porosity of the final sulphate product, $\varphi_{\tilde{\gamma}}$, namely for $c = 0$ and $\tilde{\gamma} = \frac{m_\gamma}{m_c} c_0 + \gamma_0$:

$$\varphi(c) = \varphi_{\tilde{\gamma}} + (\varphi_0 - \varphi_{\tilde{\gamma}})\,\frac{c}{c_0}. \tag{2.4}$$

We denote by s the porous concentration of SO_2, which is defined as the concentration taken with respect to the volume of the pores. The porous concentration is related to the total concentration by

$$\rho_s = \varphi(c)s. \tag{2.5}$$

At the same time, the seepage velocity $\mathbf{v}_s$ is related to the fluid velocity $\mathbf{V}_s$ by the classical Dupuit-Forchheimer relation

$$\mathbf{v}_s = \varphi(c)\mathbf{V}_s. \tag{2.6}$$

To close system (2.2), we need an expression for the seepage velocity $\mathbf{v}_s$. Following [2], we assume that all the contributions given by the pressure gradient to the seepage velocity, usually driven by the Darcy law, will be neglected. Therefore, we shall express $\mathbf{v}_s$ only for the influence due to the Fick's law [12]:

$$s\mathbf{v}_s = -D(c)\nabla s, \tag{2.7}$$

where $D(c) = d\varphi(c)$, and d is the (scalar) effective molecular diffusive coefficient. This yields, for the new unknown (s, c)

$$\begin{cases} \partial_t(\varphi(c)s) &= -\frac{k}{m_c}\varphi(c)sc + d\nabla\cdot(\varphi(c)\nabla s), \\ \\ \partial_t c &= -\frac{k}{m_s}\varphi(c)sc. \end{cases} \tag{2.8}$$

It is easy to see that we can recover the scaled model (1.1) just by restricting our attention to one space dimension and making the simple change of variables

$$y = \sqrt{\frac{k}{dm_c}}x,\ \ \tau = \frac{k}{m_c}t, \quad \tilde{s} = \frac{m_c}{m_s}s.$$

3. A priori estimates

In this section we investigate a linearized version of problem (1.1)–(1.2), which verifies some classical assumptions leading to existence and uniqueness of solutions. Let us introduce the following set of functions

$$X = \left\{ \begin{array}{l} f \in C([0,T]; L^2(\mathbb{R}))\ ,\ f_x \in L^2(\mathbb{R}\times[0,T]) : \\ \\ f(x,t) \geq 0, \quad \|f\|_{L^\infty(\mathbb{R}\times[0,T])} \leq S_0\ , \\ \\ \sup_{[0,T]}\|f\|^2_{L^2(\mathbb{R})} + \|f_x\|^2_{L^2(\mathbb{R}\times[0,T])} \leq \frac{\varphi_M}{\varphi_m}\|s_0\|^2_{L^2(\mathbb{R})} \end{array} \right\}. \tag{3.1}$$

Set also $X^\alpha = X \cap C^{2+\alpha, 1+\frac{\alpha}{2}}(\mathbb{R}\times[0,T])$ for $\alpha \in (0,1)$. In X^α, X, we consider the norm $|v|_X = \sup_{[0,T]}\|v\|^2_{L^2(\mathbb{R})} + \|v_x\|^2_{L^2(\mathbb{R}\times[0,T])}$.

Given a function $f \in X$, we consider the ordinary differential problem

$$\begin{cases} \partial_t g = -\varphi(g)gf, \\ \\ g(x,0) = c_0(x)\ , \end{cases} \tag{3.2}$$

where $0 < m \le c_0(x) \le C_0$ and $C_0 - c_0(x) \in C_0^\infty(\mathbb{R})$. The solution g can be explicitly written in terms of $\int_0^t f$, i.e.:

$$g(x,t) = \frac{Ac_0(x)}{\varphi(c_0)e^{A\int_0^t f} - Bc_0(x)} \ . \tag{3.3}$$

Let Φ be the operator which maps the set X into the set of the solutions s to the Cauchy problem for the equation

$$\partial_t(\varphi(g)s) = \partial_x(\varphi(g)\partial_x s) - \varphi(g)gs, \tag{3.4}$$

with the initial condition

$$s(x,0) = s_0(x), \tag{3.5}$$

where $s_0(x) \in C_0^\infty(\mathbb{R})$ and $0 \le s_0(x) \le S_0$.

We shall investigate some properties of the operator Φ in view of the link between its fixed points in X and the solutions of problem (1.1)–(1.2). More precisely, in this section, we shall prove that:

i) Φ is well defined on X^α;
ii) $\Phi(X^\alpha) \subseteq X^\alpha$;
iii) Φ is a contraction function in X^α.

In the following we shall denote by k_0, every constant which depends only on $A, B, \varphi_m,\ \varphi_M, S_0, C_0, m$, the L^2-norm of $s_0(x)$ and the H^1-norm of $C_0 - c_0(x)$. With $\omega_0(t)$ we shall denote quantities depending on the above constants and on t, which increase with t and go to zero if t goes to zero.

Our first step is to prove some a priori estimates for the solution g of (3.2).

Proposition 3.1. *Let $f \in X^\alpha$ for some $0 < \alpha < 1$, let g be the solution of (3.2) in $\mathbb{R} \times [0,T]$. Then for all $(x,t) \in \mathbb{R} \times [0,T]$*

$$0 < \frac{Ac_0(x)}{k_0} e^{-AS_0 t} \le g(x,t) \le C_0 \ , \tag{3.6}$$

$$\sup_{[0,t]} \|C_0 - g\|^2_{L^2(\mathbb{R})} \le 2\left(\|C_0 - c_0\|^2_{L^2(\mathbb{R})} + k_0 t^2 \sup_{[0,T]} \|f\|^2_{L^2(\mathbb{R})} \right) \ , \tag{3.7}$$

$$\sup_{[0,t]} \|g_x\|^2_{L^2(\mathbb{R})} \le \left(\|c_{0x}\|^2_{L^2(\mathbb{R})} \ + k_0 \|f_x\|^2_{L^2(\mathbb{R}\times[0,T])} \right) e^{k_0 t} \ . \tag{3.8}$$

Proof. The estimate (3.6) immediately follows from the explicit expression of g and conditions $g_t \le 0$,$c_0(x) \ge m > 0$; if $B > 0$ then $k_0 = \max_{\mathbb{R}} \varphi(c_0(x))$, otherwise $k_0 = A$.

In order to prove the second estimate we consider the equation for $\tilde{g} = C_0 - g$

$$\tilde{g}_t = \varphi(g)(C_0 - \tilde{g})f \ . \tag{3.9}$$

Multiplying by $\tilde{g}$ and integrating on $\mathbb{R} \times [0,t]$ yields

$$\int_{\mathbb{R}} \tilde{g}^2 \le \int_{\mathbb{R}} |C_0 - c_0|^2 + \varphi_M C_0 \left(\frac{1}{\epsilon} \int_0^t \int_{\mathbb{R}} f^2 + \epsilon \int_0^t \int_{\mathbb{R}} \tilde{g}^2 \right) \ , \qquad \epsilon > 0 \ .$$

For $\epsilon = \frac{1}{2\varphi_M C_0 t}$ the above inequality gives (3.7).

Finally we derive with respect to x the equation in (3.2) and then we multiply it by g_x to have

$$\|g_x\|^2_{L^2(\mathbb{R})} \le \|c_{0x}\|^2_{L^2(\mathbb{R})} + k_0 \|f_x\|^2_{L^2(\mathbb{R}\times[0,T])} + k_0 \int_0^t \int_{\mathbb{R}} g_x^2 \ ,$$

which gives (3.8). $\square$

Now we are able to prove that problem (3.4)–(3.5) has a unique classical solution.

Proposition 3.2. *Let $T > 0$ and let $f \in X^\alpha$ for some $0 < \alpha < 1$. Then, there exists a unique classical solution s in $\mathbb{R} \times [0, T]$ to problem (3.4)–(3.5), with $s \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0, T])$.*

Proof. Let us rewrite the linear parabolic equation (3.4) as follows

$$s_t = s_{xx} + \frac{Bg_x s_x}{\varphi(g)} + gs(Bf - 1) \ . \tag{3.10}$$

Notice that g_x can be explicitly expressed by

$$g_x(x,t) = \frac{c_{0x}(x,t)}{c_0^2(x)} g^2(x,t) e^{A\int_0^t f} - g(x,t)\varphi(g(x,t)) \int_0^t f_x \ . \tag{3.11}$$

We are going to prove that the coefficients $\frac{g_x}{\varphi(g)}$, $g(Bf - 1)$ are smooth.

By using the equation for g it is easy to prove that, for $(x_1, t), (x_2, t) \in \mathbb{R}$,

$$|g(x_1, t) - g(x_2, t)| \le K_1 |x_1 - x_2|,$$

where K_1 depends on φ_M, C_0 and the Lipschitz constant of f with respect to x . Moreover we know that f_x, f_{xx} and g_x are bounded in $\mathbb{R}$. By deriving the equation for g, now we are able to prove that, for $(x_1, t), (x_2, t) \in \mathbb{R}$,

(3.12)

$$|g_x(x_1, t) - g_x(x_2, t)| \le |g_{0x}(x_1, t) - g_{0x}(x_2, t)| + K_2 \left(|g(x_1, t) - g(x_2, t)| + \right.$$

$$\left. |f(x_1, t) - f(x_2, t)| + |f_x(x_1, t) - f_x(x_2, t)|\right) \le K_3 |x_1 - x_2| \ ,$$

where K_2, K_3 depend on $T, \|g_x\|_\infty, C_0, \varphi_M, S_0$ and the Lipschitz constants of f and f_x, with respect to x. It follows that the coefficient $\frac{g_x}{\varphi(c)}$ is Lipschitz continuous with respect to x in $\mathbb{R} \times [0, T]$. In similar way we prove that it is Lipschitz continuous with respect to t and we obtain the same results for the coefficient $g(Bf - 1)$. By Theorem 5.1 in [11] (pag. 320), such smoothness of the coefficients is enough to ensure the existence and uniqueness of a solution $s \in C^{2+\alpha,1+\frac{1}{\alpha}}(\mathbb{R} \times [0, T])$. □

The above result implies that the operator Φ is well defined on the sets X^α.

Next we prove some estimates for $s = \Phi(f)$, $f \in X^\alpha$; it is readily seen that they imply the property *ii*) for the operator Φ.

In the cases of increasing φ we have to consider only small initial data s_0. Let us assume that

$$\text{if } B > 0, \text{ then } S_0 < \frac{1}{B}. \tag{3.13}$$

Proposition 3.3. *Let $f \in X^\alpha$ and let assumption* (3.13) *be satisfied. Let s be the solution of problem (3.4)–(3.5); then*

$$0 \le s(x, t) \le S_0 \quad \textit{for all } (x, t) \in \mathbb{R} \times [0, T] \ ;$$

$$\lim_{x \to \pm\infty} s(x, t) = 0 \ ; \tag{3.14}$$

$$\int_{\mathbb{R}} \varphi(g) s^2 + \int_0^t \int_{\mathbb{R}} \varphi(g) s_x^2 \le \int_{\mathbb{R}} \varphi(g) s_0^2 \tag{3.15}$$

Proof. First we prove the upper bound for s. Let $\delta, \gamma > 0$, $\mu = S_0 + \delta + \frac{\gamma}{2}$. Thanks to Proposition 3.2, we can set $t_0 = \sup\{t \in [0, T] : \|s\|_{L^\infty(\mathbb{R} \times [0,T])} \le \mu\}$. Let R be a positive constant sufficiently large such that

$$\frac{2\delta}{R^2} + |B| \, \|g_x\|_{L^\infty(\mathbb{R} \times [0,T])} \frac{2\delta}{\varphi_m R} < (\min_{\mathbb{R} \times [0,T]} g)(S_0 + \gamma) \min\{1, 1 - B\|f\|_\infty\}.$$

We want to show that the function

$$w^R = s - \frac{\delta}{R^2}x^2 - S_0 - \gamma$$

is strictly negative on the domain $[-R, R] \times [0, t_0]$. On the boundary we have

$$w^R(x, 0) = s_0(x) - S_0 - \frac{\delta}{R^2}x^2 - \gamma < 0 \ ,$$

$$w^R(\pm R, t) = s(\pm R, t) - S_0 - \delta - \gamma \leq -\frac{\gamma}{2} < 0;$$

we suppose that $w^R = 0$ somewhere in $(-R, R) \times (0, t_0]$ and we denote by $P_1(x_1, t_1)$ a point such that $w^R(P_1) = 0$ and $w^R(x, t) < 0$ for $t < t_1$. Hence $w^R_t(P_1) \geq 0$, $w^R_x(P_1) = 0$ and $w^R_{xx}(P_1) \leq 0$. Taking into account the equation satisfied by w^R_1, we obtain

$$w^R_t(P_1) \leq 2\frac{\delta}{R^2} + \|g_x\|_{L^\infty(\mathbb{R}\times[0,T])}\frac{2\delta|B|}{\varphi_m R} - (\min_{\mathbb{R}\times[0,T]} g)(S_0+\gamma)\min\{1, 1-B\|f\|_\infty\} < 0$$

which leads to a contradiction.

Then $w^R < 0$ in $[-R, R] \times [0, t_0]$ and $s < S_0 + \gamma + \frac{\delta}{R^2}x^2$ on the same domain. Letting $R \to +\infty$ and then $\gamma \to 0$ we obtain $s \leq S_0$ for $(x, t) \in \mathbb{R} \times [0, t_0]$. So $t_0 = T$ and the bound is proved.

The proof of the lower bound for s follows by similar arguments. We introduce the function

$$v^R(x, t) = s(x, t) + \frac{S_0}{R^2}x^2 + \gamma$$

and we prove that if $\gamma, R > 0$ are such that

$$\frac{2S_0}{R^2} + \frac{2S_0|B|}{\varphi_m R}\|g_x\|_{L^\infty(\mathbb{R}\times[0,T])} < (\min_{\mathbb{R}\times[0,T]} g)\gamma \min\{1, 1 - B\|f\|_\infty\},$$

then $v^R(x, t) > 0$ in $[-R, R] \times [0, T]$. Letting $R \to +\infty$ and then $\gamma \to 0$ we obtain $s \geq 0$ for $(x, t) \in \mathbb{R} \times [0, T]$.

In order to prove that s goes to zero for large x, let us denote by $[c, d]$ the support of s_0 and we introduce two parameters a, R such that $a > \max\{0, d\}, R > a$ and the function

$$w^{R,a} = s - \frac{S_0 x}{R} - \frac{S_0 a}{x} - \frac{t}{\sqrt{a}}.$$

We are going to prove that $w^{R,a} < 0$ in $[a, R] \times [0, T]$. On the boundary

$$w^{R,a}(x, 0) = -\frac{S_0 x}{R} - \frac{S_0 a}{x} < 0 \ ,$$

$$w^{R,a}(a, t) = s(a, t) - \frac{S_0 a}{R} - S_0 - \frac{t}{\sqrt{a}} < 0 \ ,$$

$$w^{R,a}(R, t) = s(R, t) - S_0 - \frac{S_0 a}{R} - \frac{t}{\sqrt{a}} < 0 \ .$$

If $w^{R,a} = 0$ somewhere in $[a, R] \times [0, T]$, there exists a point $P_1 = (x_1, t_1) \in (a, R) \times (0, T]$ such that $w^R(P_1) = w^R_x(P_1) = 0, w^R_{xx}(P_1) \leq 0$ and $w^R_t(P_1) \geq 0$. Using the equation satisfied by w^R in P_1 we obtain the inequality

$$w^R_t(P_1) \leq \frac{1}{\sqrt{a}}\left(-1 + \frac{2S_0}{a^{\frac{3}{2}}} + |B|\,\|g_x\|_{L^\infty(\mathbb{R}\times[0,T])}\left(\frac{\sqrt{a}}{R} + \frac{1}{\sqrt{a}}\right)\right)$$

which, for a, R sufficiently large, leads to a contradiction.

Hence, letting $R \to +\infty$, for large a we have

$$s < \frac{S_0 a}{x} + \frac{t}{\sqrt{a}} \qquad \text{in } [a, \infty) \times [0, T];$$

it follows that

$$\lim_{x\to+\infty} s(x,t) = \frac{t}{\sqrt{a}} \qquad \text{for all } a \geq a_0$$

and then the claim for $x \to +\infty$. The claim for $x \to -\infty$ follows in similar way.

Finally we are going to prove inequality (3.15). After multiplying the equation (3.4) by s we obtain

$$\frac{(\varphi(g)s^2)_t}{2} = (\varphi(g)ss_x)_x - \varphi(g)s_x^2 + \varphi(g)gs^2\left(B\frac{f}{2} - 1\right),$$

where we used the equation for g. Now the estimate (3.15) easily follows integrating the above equality over $\mathbb{R} \times [0,T]$ since the first term on the right hand side does not give any contribute, thank to (3.14), and the last one is negative. □

Next are some estimates, which are useful in order to show that Φ verifies the contraction property *iii*).

Lemma 3.1. *Let $f_1, f_2 \in X$. Then for every fixed x and for all $t \in [0,T]$*

$$\int_0^t |f_1(x,\tau) - f_2(x,\tau)|^2 \, d\tau \leq \max\{t,1\}|f_1 - f_2|_X^2 \ .$$

Proof. By Sobolev embedding theorems we obtain

$$\begin{aligned}
&\int_0^t |f_1(x,\tau) - f_2(x,\tau)|^2 \, d\tau \leq \int_0^t \sup_{\mathbb{R}} |f_1(x,\tau) - f_2(x,\tau)|^2 \, d\tau \\
&\leq \int_0^t \left(\|f_1(\tau) - f_2(\tau)\|^2_{L^2(\mathbb{R})} + \|f_{1x}(\tau) - f_{2x}(\tau)\|^2_{L^2(\mathbb{R})}\right) \, d\tau \\
&\leq t\sup_{[0,t]} \|f_1(\tau) - f_2(\tau)\|^2_{L^2(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} |f_{1x}(x,\tau) - f_{2x}(x,\tau)|^2 \, d\tau \\
&\leq \max\{t,1\}|f_1 - f_2|_X^2 \ .
\end{aligned} \tag{3.16}$$

This proves the claim. □

Proposition 3.4. *Let g_1, g_2 be the solutions of problem (3.2) respectively with $f = f_1, f_2 \in X^\alpha$. Then for all $t \in [0,T]$*

$$\|g_1(t) - g_2(t)\|_\infty^2 \leq \omega_0(t)|f_1 - f_2|_X^2 \ , \tag{3.17}$$

$$\|g_1(t) - g_2(t)\|_2^2 \leq \omega_0(t)|f_1 - f_2|_X^2 \ , \tag{3.18}$$

$$\|g_{1x}(t) - g_{2x}(t)\|_2^2 \leq \omega_0(t)|f_1 - f_2|_X^2 \ . \tag{3.19}$$

Proof. We shall use the explicit expressions for g_1 and g_2. We observe that

$$\left(\varphi(c_0(x))e^{A\int_0^t f_1(x,\tau)d\tau} - Bc_0(x)\right)\left(\varphi(c_0(x))e^{A\int_0^t f_2(x,\tau)d\tau} - Bc_0(x)\right) \geq \varphi_m^2 \ ;$$

hence

$$|g_1(x,t) - g_2(x,t)| \leq \frac{A^2c_0(x)\varphi(c_0(x))e^{AS_0t}}{\varphi_m^2}\left|\int_0^t (f_1(x,\tau) - f_2(x,\tau))\, d\tau\right| \ .$$

Then, thanks to Lemma 3.1,

$$|g_1(x,t) - g_2(x,t)|^2 \leq k_0e^{k_0t}t\int_0^t |f_1(x,\tau) - f_2(x,\tau)|^2 d\tau \leq \omega_0(t)|f_1 - f_2|_X^2 \ .$$

The estimate (3.17) immediately follows. The above inequality also implies that

$$\int_{\mathbb{R}} |g_1(x,t) - g_2(x,t)|^2 dx \le k_0 e^{k_0 t} t^2 \sup_{[0,t]} \|f_1(t) - f_2(t)\|_2^2$$

and (3.18) is proved.

To prove the last estimate we consider the explicit expression for g_{1x} and g_{2x} and we compute

$$|g_{1x}(x,t) - g_{2x}(x,t)| = \Bigg| \frac{c_{0x}(x)}{c_0^2(x)} \left(g_1^2(x,t) e^{A\int_0^t f_1(x,\tau)d\tau} - g_2^2(x,t) e^{A\int_0^t f_2(x,\tau)d\tau} \right)$$

$$- \left(g_1(x,t)\varphi_1(x,t) \int_0^t f_{1x}(x,\tau)d\tau - g_2(x,t)\varphi_2(x,t) \int_0^t f_{2x}(x,\tau)d\tau \right) \Bigg|$$

where $\varphi_i = \varphi(g_i)$, $i = 1,2$. Then we have

(3.20)

$$|g_{1x}(x,t) - g_{2x}(x,t)|^2 \le k_0 |c_{0x}(x)|^2 e^{2At} \left| \int_0^t (f_1(x,\tau) - f_2(x,\tau)) d\tau \right|^2$$

$$+ k_0 |c_{0x}(x)|^2 e^{2At} |g_1(x,t) - g_2(x,t)|^2 + k_0 g_1^2(x,t) \left(\int_0^t |f_{2x}(x,\tau) - f_{1x}(x,\tau)| d\tau \right)^2$$

$$+ k_0 \left(\int_0^t |f_{2x}(x,\tau)| d\tau \right)^2 |g_1(x,t) - g_2(x,t)|^2 \ ,$$

therefore

$$\begin{aligned} \|g_{1x}(t) - g_{2x}(t)\|_2^2 &\le \omega_0(t) \|c_{0x}\|_2^2 \int_0^t \sup_{\mathbb{R}} |f_1(\tau) - f_2(\tau)|^2 \ d\tau \\ &+ k_0 e^{2At} \|c_{0x}\|_2^2 \, \|g_1(t) - g_2(t)\|_\infty^2 \qquad (3.21) \\ &+ \omega_0(t) \int_{\mathbb{R}} \int_0^t |f_{1x} - f_{2x}|^2 + \omega_0(t) \|g_1(t) - g_2(t)\|_\infty^2 \int_0^t \int_{\mathbb{R}} |f_{2x}|^2. \end{aligned}$$

Now, using Lemma 3.1, (3.17) and (3.18) we obtain (3.19). ☐

Now we are going to prove that, for small T, the operator Φ is a contraction function in the norm $|\cdot|_X$ on X^α.

Proposition 3.5. *Let assumption* (3.13) *be verified and let* $f_1, f_2 \in X^\alpha$ *and* $s_1 = \Phi(f_1), s_2 = \Phi(f_2)$. *Then there exists a constant* $L < 1$ *such that, for* $T \le T_0$,

$$|s_1 - s_2|_X \le L |f_1 - f_2|_X \ ,$$

where $T_0 = T_0(A, B, \varphi_m, \varphi_M, S_0, C_0, m, \|s_0\|_{L^2(\mathbb{R})}, \|c_{0x}\|_{L^2(\mathbb{R})})$.

Proof. We set $\Sigma = s_1 - s_2$, $\Gamma = g_1 - g_2$ and we consider the equation (3.4) for s_1 and s_2. Subtracting the equation for s_2 to the equation for s_1 and multiplying by $(s_1 - s_2)$, we find

$$\begin{aligned} \left(\varphi_1 \tfrac{\Sigma^2}{2}\right)_t + B g_t \tfrac{\Sigma^2}{2} \quad &+ B\left(s_2 \Gamma \Sigma\right)_t - B s_2 \Gamma \Sigma_t \qquad (3.22) \\ &= (\varphi_1 s_{1x} - \varphi_2 s_{2x})_x \Sigma + (\varphi_1 g_1 s_1 - \varphi_2 g_2 s_2)\Sigma. \end{aligned}$$

Now we integrate over $\mathbb{R} \times [0,t]$ and using the equation for g and (3.14) we have

$$
\begin{aligned}
\int_{\mathbb{R}} \varphi_1 \frac{\Sigma^2}{2}\, dx \;&=\; B \int_0^t \int_{\mathbb{R}} g_1 \varphi_1 f_1 \frac{\Sigma^2}{2}\, dx\, d\tau \\
&-B \int_{\mathbb{R}} s_2(t)\Gamma(t)\Sigma(t)\, dx + B \int_0^t \int_{\mathbb{R}} s_2 \Gamma \Sigma_t\, dx\, d\tau \\
&- \int_0^t \int_{\mathbb{R}} (\varphi_1 s_{1x} - \varphi_2 s_{2x})\Sigma_x\, dx\, d\tau + \int_0^t \int_{\mathbb{R}} (\varphi_1 g_1 s_1 - \varphi_2 g_2 s_2)\Sigma\, dx\, d\tau\ .
\end{aligned} \tag{3.23}
$$

We are going to obtain some estimates for the terms on the right hand side using Proposition 3.4, the L_∞ bounds for s_i, f_i, the bounds (3.6)-(3.8) for g_i and the boundedness of $|s_i|_X$ and $|f_i|_X$, for $i = 1, 2$.

For the first term we have

$$
|B| \int_0^t \int_{\mathbb{R}} g_1 \varphi_1 f_1 \frac{\Sigma^2}{2}\, dx\, d\tau \le \omega_0(t) \sup_{[0,t]} \int_{\mathbb{R}} \varphi_1 \frac{\Sigma^2}{2}\, dx
$$

and for the second one

$$
|B| \int_{\mathbb{R}} s_2(t) |\Gamma(t)|\, |\Sigma(t)|\, dx \le \frac{\omega_0(t)}{\epsilon} |f_1 - f_2|_X^2 + \epsilon \|\Sigma\|_2^2\ , \qquad \epsilon > 0\ .
$$

Next, for the fourth term, we have

$$
\begin{aligned}
&- \int_0^t \int_{\mathbb{R}} (\varphi_1 s_{1x} - \varphi_2 s_{2x})\Sigma_x\, dx\, d\tau \\
&= - \int_0^t \int_{\mathbb{R}} \varphi_1 (\Sigma_x)^2\, dx\, d\tau - B \int_0^t \int_{\mathbb{R}} s_{2x} \Gamma \Sigma_x\, dx\, d\tau \\
&\le - \int_0^t \int_{\mathbb{R}} \varphi_1 (\Sigma_x)^2\, dx\, dt + \frac{k_0}{2\epsilon} \|\Gamma\|_\infty^2 \int_0^t \int_{\mathbb{R}} |s_{2x}|^2\, dx\, d\tau + \frac{\epsilon}{2} \int_0^t \int_{\mathbb{R}} |\Sigma_x|^2\, dx\, d\tau \\
&\le \tfrac{\omega_0(t)}{\epsilon} |f_1 - f_2|_X - \left(\varphi_m - \tfrac{\epsilon}{2}\right) \int_0^t \int_{\mathbb{R}} |\Sigma_x|^2\, dx\, d\tau\ ,
\end{aligned} \tag{3.24}
$$

for small $\epsilon > 0$; in a similar way

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} (\varphi_1 g_1 s_1 - \varphi_2 g_2 s_2)\Sigma\, dx\, d\tau \\
&\le \omega_0(t) \sup\nolimits_{[0,t]} \int_{\mathbb{R}} \varphi_1 \Sigma^2\, dx + \frac{\omega_0(t)}{\epsilon} \sup_{[0,t]} \|\Gamma\|_2^2 + \omega_0(t)\epsilon \sup_{[0,t]} \|\Sigma\|_2^2 \\
&\le (1+\epsilon)\omega_0(t) \sup\nolimits_{[0,t]} \|\Sigma\|_2^2 + \tfrac{\omega_0(t)}{\epsilon} |f_1 - f_2|_X^2 \qquad \epsilon > 0\ .
\end{aligned} \tag{3.25}
$$

Finally we rewrite the third term discarding $(s_1 - s_2)_t$ by using the equation (3.4) for s_1 and s_2 and, after an integration by part of the term with the second derivatives in x, we obtain

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}} s_2 \Gamma \Sigma_t\, dx\, d\tau &= - \int_0^t \int_{\mathbb{R}} s_{2x} \Gamma \Sigma_x\, dx\, d\tau - \int_0^t \int_{\mathbb{R}} s_2 \Gamma_x \Sigma_x\, dx\, d\tau \\
&+ \int_0^t \int_{\mathbb{R}} s_2 \Gamma \frac{\varphi_2 g_{1x} s_{1x} - \varphi_1 g_{2x} s_{2x}}{\varphi_1 \varphi_2}\, dx\, d\tau + \int_0^t \int_{\mathbb{R}} s_2 s_1 (B f_1 - 1)\Gamma^2\, dx\, d\tau \\
&+ \int_0^t \int_{\mathbb{R}} s_2 g_2 \Gamma \Sigma (B f_1 - 1)\, dx\, d\tau + B \int_0^t \int_{\mathbb{R}} s_2^2 g_2 \Gamma (f_1 - f_2)\, dx\, d\tau\ ;
\end{aligned} \tag{3.26}
$$

therefore

$$
\begin{aligned}
&B\int_0^t\int_{\mathbb{R}} s_2\Gamma\Sigma_t\,dx\,d\tau \le \frac{k_0}{\epsilon}\int_0^t\int_{\mathbb{R}}|s_{2x}|^2|\Gamma|^2\,dx\,d\tau + k_0\epsilon\int_0^t\int_{\mathbb{R}}\Sigma_x^2\,dx\,d\tau \\
&+\tfrac{k_0}{\epsilon}\int_0^t\int_{\mathbb{R}}\Gamma_x^2\,dx\,d\tau + k_0\epsilon\int_0^t\int_{\mathbb{R}}\Sigma_x^2\,dx\,d\tau + k_0\int_0^t\int_{\mathbb{R}} s_2|\Gamma|\frac{|g_{1x}|}{\varphi_1}|\Sigma|\,dx\,d\tau \\
&+k_0\int_0^t\int_{\mathbb{R}} s_2|s_{2x}|\,|\Gamma|\left|\frac{g_{1x}}{\varphi_1}-\frac{g_{2x}}{\varphi_2}\right|\,dx\,d\tau + k_0\int_0^t\int_{\mathbb{R}}|\Gamma|^2\,dx\,d\tau \\
&+k_0\int_0^t\int_{\mathbb{R}}|\Sigma|^2\,dx\,d\tau + k_0\int_0^t\int_{\mathbb{R}}|f_1-f_2|^2\,dx\,d\tau \le k_0\epsilon\int_0^t\int_{\mathbb{R}}|\Sigma_x|^2\,dx\,d\tau \\
&+\|\Gamma\|_\infty^2\left(\tfrac{k_0+\omega_0(t)}{\epsilon}+k_0+\omega_0(t)\right)+\left(\tfrac{\omega_0(t)}{\epsilon}+\omega_0(t)\right)\sup_{[0,t]}\|\Gamma_x\|_2^2 \\
&+\omega_0(t)\left(\sup_{[0,t]}\|\Gamma\|_2^2+\sup_{[0,t]}\|\Sigma\|_2^2+\sup_{[0,t]}\|f_1-f_2\|_2^2\right)\ .
\end{aligned}
\tag{3.27}
$$

Now, using Proposition 3.4 yields

$$
\begin{aligned}
&B\int_0^t\int_{\mathbb{R}} s_2\Gamma\Sigma_t\ dx\ d\tau \le k_0\epsilon\int_0^t\int_{\mathbb{R}}|\Sigma_x|^2+\omega_0(t)\sup_{[0,t]}\|\Sigma\|_2^2 \\
&+\omega_0(t)\left(1+k_0+\omega_0(t)+\tfrac{k_0+\omega_0(t)}{\epsilon}\right)|f_1-f_2|_X^2\ .
\end{aligned}
\tag{3.28}
$$

Putting together all the estimates obtained for the terms on the right hand side of (3.23) we proved that

$$
\begin{aligned}
&\|\Sigma\|_2^2\left(\tfrac{\varphi_m}{2}-\omega_0(t)-\epsilon\omega_0(t)\right)+(k_0-k_0\epsilon)\int_0^t\int_{\mathbb{R}}|\Sigma_x|^2 \\
&\le |f_1-f_2|_X^2\left(\tfrac{\omega_0(t)}{\epsilon}+\omega_0(t)\right)\ .
\end{aligned}
\tag{3.29}
$$

Now it is easy to verify that for suitable small ϵ and t, there exists $L<1$ such that

$$|\Sigma|_X^2 \le L\|f_1-f_2|_X^2\ ,$$

which concludes the proof. □

4. Existence and uniqueness of solutions

Now we return to deal with problem (1.1). We already pointed out in the Introduction that our system is not a strongly parabolic problem. For this reason we introduce the following notion of weak solution.

Definition 4.1. *A pair (s,c) is a weak solution to system (1.1)–(1.2) in $\mathbb{R}\times[0,T)$ if*

a) $s\in \mathrm{L}^\infty(\mathbb{R}\times[0,T))\cap C([0,T);L^2(\mathbb{R}))\ ,\ s_x\in L^2(\mathbb{R}^+\times[0,T])$;

b)

$$\int_0^T\int_{\mathbb{R}}\left(\varphi(c)s\psi_t-\varphi(c)s_x\psi_x-\varphi(c)cs\psi\right)\ dx\,dt+\int_{\mathbb{R}}\varphi(c_0)\psi(x,0)\ dx=0$$

for all $\psi\in C_0^1(\mathbb{R}\times[0,T))$;

c) *c is given in terms of $\int_0^t s(x,\tau)d\tau$ by the expression*

$$c(x,t) = \frac{Ac_0(x)}{\varphi(c_0)e^{A\int_0^t s} - Bc_0(x)} \ . \tag{4.1}$$

We are going to show that for all $T > 0$ problem (1.1)–(1.2) has a unique weak solution (s,c) on $\mathbb{R} \times [0,T)$.

Theorem 4.1. *Let s_0, c_0 be nonnegative functions such that $0 \le s_0 \le S_0$, $0 < m \le c_0(x) \le C_0$ and $s_0 \in L^2(\mathbb{R})$, $C_0 - c_0 \in H^1(\mathbb{R})$; moreover assume that assumption (3.13) is satisfied. Then there exists $T > 0$ such that problem (1.1)–(1.2) has a unique local weak solution (s,c) on the time interval $[0,T)$, with the following properties*

i) $0 < c(x,t) \le C_0$*;*
ii) $0 \le s(x,t) \le S_0$ *in* $\mathbb{R} \times [0,T)$*;*
iii) $C_0 - c \in C([0,T); H^1(\mathbb{R}))$*;*
iv) $\sup_{[0,T)} \|s(t)\|^2_{L^2(\mathbb{R})} + \|s\|^2_{L^2(\mathbb{R}\times[0,T))} \le \frac{\varphi_M}{\varphi_m}\|s_0\|^2_{L^2(\mathbb{R})}$.

If $[0,T_M)$ is the maximal interval of existence, then $T_M < +\infty$ implies

$$\lim_{t\to T_M^-} \|c_x(\cdot,t)\|_{L^2(\mathbb{R})} = +\infty. \tag{4.2}$$

Proof. The contraction property of Φ in X^α, proved in the previous section, implies that the sequence

$$s_0(x,t) = f(x,t) \in X^\alpha \ , \qquad s_{n+1}(x,t) = \Phi(s_n(x,t))$$

for $(x,t) \in \mathbb{R} \times [0,T]$, T small, converges in $X-$norm to a function $s \in X$ such that $0 \le s(x,t) \le S_0$ and $|s|^2_X \le \frac{\varphi_M}{\varphi_m}\|s_0\|^2_2$.

By density arguments it follows that Φ is well defined on X and it is a contraction on the same set. So the limit function s is the unique fixed point of Φ in X.

Hence we proved the existence and uniqueness of the weak solution to (1.1)–(1.2) in $\mathbb{R} \times [0,T)$ for smooth initial data and T small. By density arguments we prove the same result for general initial data.

Proposition 3.5 shows that T depends on $A, B, \varphi_m, \varphi_M, S_0, C_0, m, \|s_0\|_{L^2(\mathbb{R})}$, $\|c_{0x}\|_{L^2(\mathbb{R})}$. We know that $\|s(t)\|_{L^2(\mathbb{R})}$ is bounded by a constant depending only on the data; therefore, if $\sup_{[0,T]} \|c_x(t)\|_{L^2(\mathbb{R})} \le C$, we can reconsider our system with initial data $s(x,T), c(x,T)$ and, arguing as in Section 3, we extend the solution on an interval $[0, T+\delta)$ where δ depends on $A, B, \varphi_m, \varphi_M, S_0, C_0, m, \|s_0\|_{L^2(\mathbb{R})}$ and on C. □

Now we use the continuation principle of the above theorem to prove the following global result.

Theorem 4.2. *Under the same assumptions of Theorem 4.1, we have $T_M = \infty$ and the solution is globally defined.*

Proof. Let (s,c) be the local weak solution of problem (1.1)–(1.2) and let $[0,T_M)$ be the maximal interval of existence. By Proposition 3.1 we have the following estimate

$$\sup_{[0,t]} \|c_x\|^2_{L^2(\mathbb{R})} \le \left(\|c_{0x}\|^2_{L^2(\mathbb{R})} + k_0\|s_x\|^2_{L^2(\mathbb{R}\times[0,T])} \right) e^{k_0 t} \ .$$

Hence, by the continuation principle in Theorem 4.1 we deduce that T_M cannot be finite, and then the solutions are globally defined. □

By using classical regularity results we are also able to precise the actual smoothness of our weak solutions.

Theorem 4.3. *Let s_0, c_0 satisfy conditions (1.3) and assumption* (3.13) *and let the pair (s, c) be the global weak solution of (1.1)–(1.2). There exists $\alpha > 0$ such that, if $c_0 \in C^{1+\alpha}(I)$ for all intervals $I \subset \mathbb{R}$ then*

$$s \in C^{2+\alpha,1+\frac{\alpha}{2}}(I \times (0,T))$$

for all $I \subset \mathbb{R}$ and all $T > 0$. Moreover in $I \times (0,T)$ c_x is Holder continuous with exponent α with respect to x and Lipschitz continuous with respect to t ; c_t is Holder continuous with exponent $1 + \alpha$ with respect to x and Lipschitz continuous with respect to t.

Proof. We are going to use Theorem 12.1 in [11] (Chapter III), which gives a general regularity result for linear equations with discontinuous coefficients. Let us write the parabolic equation for s as

$$s_t = s_{xx} + \frac{c_x}{\varphi(c)} s_x + cs(Bs - 1)$$

considering $\frac{c_x}{\varphi(c)}$ and $c(Bs-1)$ as coefficients depending on (x,t). Next, we introduce the following norms

$$\|v\|_{q,r,I,T} = \left(\int_0^T \left(\int_I |v|^q dx \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}} .$$

We have to verify that our coefficients have finite norms $\|\cdot\|_{2q,2r,I,T}$ for some positive q, r satisfying the conditions

$$\begin{aligned} &\tfrac{1}{r} + \tfrac{1}{2q} = 1 - \theta \ , \\ &q \in [1, +\infty] \ , \quad r \in \left[\tfrac{1}{1-\theta}, \tfrac{2}{1-2\theta} \right] \ , \quad 0 < \theta < \tfrac{1}{2} \ . \end{aligned} \tag{4.3}$$

Thanks to (3.8) and the L^∞ bounds for s and c it is easy to check that

$$\begin{aligned} &\left\| \tfrac{c_x}{\varphi(c)} \right\|_{1,6,I,T} \leq C_1(T) \ , \\ &\|c(Bs-1)\|_{1,6,I,T} \leq C_2(I,T) \ , \end{aligned} \tag{4.4}$$

where $C_1(T)$ and $C_2(I,T)$ are positive constants; $q = 1$ and $r = 6$ satisfy conditions (4.3) with $\theta = \frac{1}{3}$.

Inequalities (4.4) imply not only that, for some $\alpha > 0$ depending on q, r, the solution $s \in C^{\alpha,\frac{\alpha}{2}}(I \times (0,T))$, but also that s_x belongs to $C^{\alpha,\frac{\alpha}{2}}(I \times (0,T))$. Now we can use this information to prove more regularity for the coefficients. In fact, as in Proposition 3.2, it easy to verify that $\frac{c_x}{\varphi(c)}, c(Bs-1) \in C^{\alpha,\frac{\alpha}{2}}(I \times (0,T))$, using the expressions of c, c_x in function of s, the smoothness of s and assumption on c_{0x}. The claim follows by using again Theorem 12.1 in [11]. □

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