# ON RELAXATION HYPERBOLIC SYSTEMS VIOLATING THE SHIZUTA–KAWASHIMA CONDITION

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Abstract. In this paper, we start a general study on relaxation hyperbolic systems which violate the Shizuta–Kawashima coupling condition ([SK]). This investigation is motivated by the fact that this condition is in general not satisfied by various physical systems, and almost all the time in several space dimensions. First, we explore the rôle of entropy functionals around equilibrium solutions, which may be not constant, proposing a stability condition for such solutions. Then we find strictly dissipative entropy functions for one dimensional $2 \times 2$ systems which violate [SK] condition. Finally, we prove the existence of global smooth solutions for a class of systems such that condition [SK] does not hold, but which are linearly degenerated in the non dissipative directions.

Dissipative hyperbolic systems, stability conditions, entropy functional, global existence of solutions

## 1. Introduction

**Prologue.** A hyperbolic system with relaxation is a particular type of hyperbolic system of balance laws, presenting a two scales dynamic determined by the presence of a relaxation term with a characteristic time $\varepsilon$: for short times, the behavior is mainly determined by the interactions between hyperbolic propagation and relaxations effect, which drive the system toward a given equilibrium manifold; for large times, a relaxed structure, which under suitable assumptions is described by a reduced diffusive system of conservation laws, emerges determining the main features of the asymptotic behavior of the solution. The main challenge in the mathematical analysis of this kind of systems is to understand the interaction between hyperbolic convective/transport effects and zero order dissipative terms.

Many physical models fit into this framework, the prototype of them being the compressible Euler system for isentropic flows with damping, see [31, 18, 35], where the usual equations of gasdynamics are coupled with a frictional term describing the presence of dissipation effects in the dynamics. Here we consider the Cauchy problem for a general hyperbolic $N$-dimensional system of balance laws

$$w_t + \operatorname{div}_x F(w) = \frac{1}{\varepsilon} G(w). \tag{1}$$

where $F = (F_1, \dots, F_d)$ and $G$ are assumed to be smooth functions. This system is supplemented by initial conditions

$$w(x, 0) = w_0(x). \tag{2}$$

Besides the large amount of papers dealing with numerical aspects, the analytical results in the literature on hyperbolic systems with relaxation can be divided in two main subsets, the two directions being related, but somewhat complementary:

– *(Singular limits)* For a fixed time interval, one studies the limiting behavior as the temporal scale of the relaxation dynamics tends to zero, i.e. $\varepsilon \to 0^+$. The main goal of this singular perturbation problem is to prove rigorously the continuity with respect to the fast time scale in order to justify the reduction obtained by considering istantaneous equilibrium. Along this direction, we refer the reader to [30] for a review of results before 1998, and to [33, 2, 3] for some more recent interesting results.

– *(Stability/Large time behavior)* For a fixed relaxation time scale $\varepsilon$, one considers the solutions' behavior as time goes on, paying particular attention to formation of singularities/smoothness and the asymptotic behavior of solutions, i.e. $t \to +\infty$. The presence of a dissipating source term can compensate the loss of regularity due to the quasilinear hyperbolic structure of the system – which is usually able to generate gradient catastrophies in finite time in the homogeneous case – and global smooth solutions may exist. The easiest example of this competition is given by a zero order term describing a kinetic mechanism with a single global attracting state. This case is considered in the Appendix and it should be seen as a good first approach for beginners. Nevertheless, in physical systems only partial dissipation is present. The first evidence of this is the presence of an equilibrium manifold, that is a manifold of equilibrium states with dimension $n \geq 1$. When the system satisfies appropriate coupling conditions (details will be given later on), the partial dissipation term may produce complete viscous effects. In this case, by means of a Chapman–Enskog expansion, it is possibile to show that the hyperbolic system with relaxation behaves in a way similar to some appropriate (reduced) systems of viscous conservation laws [25, 10]. As a consequence, under appropriate assumptions, stability of constant states holds, in the sense that small perturbations to a given state in the equilibrium manifold give raise to global smooth solutions, possibly converging asymptotically to the unperturbed state as $t \to +\infty$ (among others, see [13, 12, 11, 16, 32, 35, 4]).

In this second line of research, two main aspects are present:
— constant steady states are particular permanent (i.e. defined for any $t \in \mathbb{R}$) smooth solutions and their regularity is inherited by sufficiently small perturbations around them. Let us remark that, if relaxation is absent, the same property does not hold in general and shock may occur in finite time.;
— constant steady states are attracting solution and the rate of decay, usually determined either by Green function estimates or by analysis of asymptotic profiles (diffusion waves), is the same of the heat kernel.

Similar properties hold also in the case of relaxation shock profiles, see [28, 29] and [27] for a recent review on the subject, and for rarefaction waves [24].

The main strategy in proving stability of constant states is based on energy estimates, i.e. in the Sobolev space $H^s$ with appropriate $s$. Following the analysis in [26] for hyperbolic-parabolic systems, coupling energy estimates with pointwise estimates of the Green function of the linearized problem permits to prove very detailed and general results, under a very specific conditions called Shizuta–Kawashima condition, see [34, 4] and references therein. Again, the link between the two classes of nonlinear evolution equations (hyperbolic and parabolic) is the Chapman–Enskog expansion (for details, see [38, 4]).

The starting step in the energy estimates approach is to control the $L^2$ norm of the perturbation, i.e. to determine an invariant set in $L^2$. At this level, the basic tool to work with is the entropy $\mathcal{E}$, a convex function of the state variable $w$ satisfying appropriate compatibility relations with the flux function $F$. Convexity (in the strict sense) guarantees that the entropy estimate is equivalent to the $L^2$ estimate. In order to have invariance or, in other words, to guarantee that $\mathcal{E}$ is a Lyapunov functional for the system, the entropy has to be dissipative. Even if entropy does not exist for general systems and, when it does, it could be not dissipative, many physical systems possess such a functional. Hence, many contributions in the literature are based on the assumption of existence of a dissipative entropy and many stability conditions based on entropy have been analyzed, [5, 10, 16, 22, 38, 32].

Once the $L^2$-estimate has been obtained, the next step is to determine estimates on higher order derivatives. Again using entropies (precisely, linearization of the entropy at the constant state) it is possible to obtain estimates analogous to the 0-th order one, but with some additional term. Since the dissipation appears in the systems only relatively to the relaxation mechanism, the estimates do not close. At this point, the key assumption, namely Shizuta–Kawashima condition, enters in the game, see Condition [SK] below for a precise statement of this condition. This hypothesis can be read in many different ways. In term of stability, it guarantees the necessary coupling between conserved/non conserved quantities in order to have dissipation effects in both the state variables. Additional energy estimates, based on condition [SK], permit to close the analysis and to prove (asymptotic) stability.

**Aim of the present paper.** Existence of dissipative entropies plus the Shizuta–Kawashima condition guarantee stability. Some models satisfy the two requests (among others, as already mentioned, the Euler equation for isentropic gases with damping, and the relaxation and BGK systems proposed in [20, 1].) Nevertheless, there are many physical systems, especially in the multidimensional case, still possessing dissipative entropies, for which assumption [SK] does not hold. An interesting example is the model for gas dynamics in thermal non equilibrium considered in [39], where stability of constant states has been proved. Here, stability is not asymptotical: small initial perturbations give raise to global smooth solutions that do not decay, but they stay small uniformly with respect to the time variable $t \in (0, \infty)$. Some other models share the same lack of dissipativity (translating in the fact that [SK] does not hold): as for instance, the Kerr–Debye model for electromagnetic waves in nonlinear Kerr medium [15, 8, 9], the hierarchy of equations coming from the kinetic formulation of multi-branch entropy solutions of scalar conservation laws [6, 7], and the very simple example on traffic flow in [23].

Hence, an interesting direction of investigation is to determine weaker conditions, in principle partial versions of [SK], guaranteeing stability, at least around steady states. The present paper aims to give a contribution in this direction from a general/structural point of view. We stress once more that all of the analysis concerns with *smooth* solution. Weak formulation, needed to deal with possibly discontinuos solutions is never taken in account in what follows.

Our point of view is based on the following question:

> *how far is it possible to go in the stability analysis* <u>*without*</u> *assuming the Shizuta–Kawashima condition?*

Two main new features appear when going in this direction:

– *(possible) existence of non constant equilibrium solutions.* The set of equilibrium solutions, i.e. solutions with values in the equilibrium manifold, can be much richer than in the case when condition [SK] is satisfied. This phenomenon, corresponding to the presence of persistent propagating signals, prevents asymptotic stability of constant states and it forces to look for stability results alone;

– *necessity of linear degeneration to prevent shock formation.* Because of weaker dissipation, existence of global smooth solutions is related to the presence of some linear degeneracy of the reduced equation in the non-dissipated direction. The most trivial example in this sense is given by a system composed by two decoupled subsystems, the first of the two being linear and the second totally dissipative.

We are interested in determining minimal conditions assuring stability of such equilibrium solutions and in making clear the rôle of linear degeneration in preventing shock formation phenomenon.

First of all, we introduce the notion of equilibrium solution, or maxwellian, for the system of balance laws (1). This is simply a solution with range in the equilibrium manifold. When the Shizuta–Kawashima condition is satisfied, any equilibrium solution is constant; while, in the general case, nonconstant equilibrium solutions may exist. In Section 3, we propose some stability conditions extending the known ones for the case of constant solutions. Since such conditions are based on the existence of a dissipative entropy, they are in $L^2$ if and only if the entropy is strictly convex (in the sense that its hessian matrix is positive definite). In the constant case, the dissipativity is related only with the coupling between the Jacobian of the entropy and the source term. In the non-constant case, also the second derivatives of the flux function are taken into account, see Condition [St] in Section 3. In practice our condition can be seen as a linear degeneracy condition of the flux functions in the directions where condition [SK] is not verified.

Convexity in the strict sense for the entropy is related with coupling between conservative/non conservative variables and the Shizuta–Kawashima condition. This is investigated in details in Section 4 in the $2 \times 2$ case. There, we show that a *strictly* convex dissipative exists even in the case where [SK] does not hold if and only if the coupling between conserved/non conserved quantity is sufficiently weak. The other way round, if the coupling is strong, then convex entropies exists if and only if the condition [SK] is satisfied.

Based on the analysis of the previous Sections, in Section 5, we build up a somewhat artificial class of models for which $H^s$ estimates (actually $H^2$, since we restrict to the one-dimensional case for the space variable $x$) can be completed. The key point is the presence of linear degeneration in the non-dissipative directions. Even if these systems are not physical, we regard them as toy models useful in making a step forward in understanding the whole picture. For instance all the characteristic fields for the Kerr–Debye model are linearly degenerate, see [8, 9], but the problem of the global existence of stable solutions is still open.

Actually, physical models usually satisfies a weak form of assumption [SK] and, in our opinion, general results should be obtained by weaker estimates coming from the Shizuta–Kawashima condition appropriately combined with the point of view of the present paper, maybe using in an appropriate way better decay properties in several space dimensions. This is left for future research. Here, let us just point out a quite different point of view followed in some recent works by Kawashima and his group, see [19, 21, 17]. Also in these investigations, some systems violating [SK] condition where considered, as for instance the dissipative Timoshenko system. However the degeneracy of the [SK] condition is of a different type with respect to the examples considered here. In particular, at the linear level, a slower decay of the Fourier transform was always taken into account, while in our examples there are directions where there is no decay at all.

## 2. Equilibrium solutions and Shizuta–Kawashima condition

Let us consider a system of balance laws for the unknown $w(x,t) = (u(x,t), v(x,t)) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$, with $(x,t) \in \mathbb{R}^{d+1}$,

$$\left\{ \begin{array}{l} u_t + \operatorname{div}_x f(u,v) = 0, \\ v_t + \operatorname{div}_x g(u,v) = q(u,v) \end{array} \right. \tag{3}$$

where $f = (f_1, \dots, f_d)$, $g = (g_1, \dots, g_d)$ and $q$ are assumed to be smooth functions. The set $\mathcal{V} := \{(u,v) \in \mathbb{R}^N \,:\, q(u,v) = 0\}$ is the equilibrium manifold of (3). Following [10, 16], we consider the case of an equilibrium manifold $\mathcal{V}$ given by a smooth $n-$dimensional manifold. More precisely we assume:

[H] $q(u,v) = 0$ if and only if $v = h(u)$ where $h : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^{N-n}$ is a smooth function ($\mathcal{U}$ is an open subset of $\mathbb{R}^n$) and

$$\operatorname{Re} \sigma(\mathrm{d}_v q(u, h(u))) \subset \{\lambda \in \mathbb{C} \,:\, \operatorname{Re} \lambda < 0\}, \tag{4}$$

where $\sigma(\cdot)$ is for the spectrum of an operator.

In the following, we can also use the shorter form

$$w_t + \operatorname{div}_x F(w) = G(w), \tag{5}$$

where $F = (f, g)$ and G=$(0_n, q)$.

Assumption [H] changes radically the situation with respect to the totally dissipative case, which will be described in the Appendix. Indeed, while in that case there is a unique isolated equilibrium state globally attractive with respect to the underlying kinetic, under [H] any equilibrium state is not isolated. For this reason, considering the stability property of a given equilibrium, it is necessary to take in account all of the solutions of (5) close to the equilibrium and lying in the equilibrium manifold.

*A function $W = W(x,t)$ is an* equilibrium solution *(or a* maxwellian*) of (5) if it lies in the equilibrium manifold $\mathcal{V}$, i.e. $G(W(x,t)) = 0$ for any $(x,t)$.*

Obviously, any constant $\overline{U} \in \mathcal{V}$, i.e. satisfying $G(\overline{U}) = 0$, is an equilibrium solution.

Consider the Jin-Xin system introduced in [20]

$$\left\{ \begin{array}{l} u_t + v_x = 0, \\ v_t + a^2 u_x = h(u) - v. \end{array} \right. \tag{6}$$

where $u, v \in \mathbb{R}^n$, $n \geq 1$. A well-known stability condition is the so-called subcharacteristic stability condition, [37, 25, 33, 16], which reads: the matrix $a^2 I - \mathrm{d}h(u)^2$ is positive definite for any $u$ under consideration. Looking for equilibrium solutions $W = (U, V)$, i.e. solutions such that $V = h(U)$, we get

$$U_t + h(U)_x = 0, \qquad \left[a^2 I - dh(U)^2\right] U_x = 0. \tag{7}$$

If $a^2 I - dh(U)^2$ is strictly positive, it is also invertible and any equilibrium solutions is constant. If we have only $a^2 - dh(U)^2 \geq 0$, then nonconstant equilibrium solutions may exist. An explicit example of the latter situation is for $n = 1$, $a = 1$, $h(u) = u$. In this case, $U = V = \phi(x - t)$ are solutions of the systems for any choice of the function $\phi$.

Existence/nonexistence of nonconstant maxwellian is strictly connected with Shizuta–Kawashima condition. This condition is obtained from (5) after linearization at a given equilibrium state $\overline{W} \in \mathcal{V}$:

$$w_t + \sum_{i=1}^{d} \mathrm{d}F_i(\overline{W})\, w_{x_i} = \mathrm{d}G(\overline{W})\, w. \tag{8}$$

The Shizuta–Kawashima condition has many equivalent formulation (see [34]). We use the following:

**Shizuta–Kawashima condition** `[SK]`. *If $z \in \ker \mathrm{d}G(\overline{W})$, then there is no $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d \setminus \{0\}$ such that $z$ is an eigenvector of the $n \times n$-matrix $\sum_{i=1}^{d} \mathrm{d}F_i(\overline{W})\, \omega_i$.*

On the contrary, assume that the Shizuta–Kawashima condition is not satisfied, i.e. assume that there exist $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d \setminus \{0\}$ and $r \in \mathbb{R}^n \setminus \{0\}$ such that

$$\mathrm{d}G(\overline{W}) r = 0 \qquad \text{and} \qquad \sum_{i=1}^{d} \mathrm{d}F_i(\overline{W})\, \omega_i\, r = \lambda\, r$$

for some $\lambda \in \mathbb{R}$. Then the linear system (8) has maxwellian of the form $w(x,t) = h(\omega \cdot x - \lambda t)\, r$. Indeed $w_t = -\lambda\, h'\, r$, $w_{x_i} = \omega_i\ h'\, r$ and

$$w_t + \sum_{i=1}^{d} \mathrm{d}F_i(\overline{W})\, w_{x_i} - \mathrm{d}G(\overline{W})\, w = h' \left( \sum_{i=1}^{d} \mathrm{d}F_i(\overline{W})\, \omega_i - \lambda\, I \right) r = 0,$$

where $h : \mathbb{R} \to \mathbb{R}$ is an arbitrary differentiable function.

In order to estabilish a converse of the previous property, we restrict our attention to the one–dimensional case: $d = 1$. Assume that $w$ is a maxwellian of (8):

$$w_t + \mathrm{d}F(\overline{W})\, w_x = 0 \qquad \text{and} \qquad \mathrm{d}G(\overline{W})\, w = 0.$$

If $r_1, \dots, r_N$ are linearly independent right eigenvectors of $\mathrm{d}F(\overline{W})$, the solution $w$ can be written as

$$w(x,t) = \sum_{i=1}^{N} k^i(x - \lambda_i t)\, r_i, \tag{9}$$

where $k^i : \mathbb{R} \to \mathbb{R}$ are the coordinates of $w$ with respect to the basis $\{r_1, \dots, r_n\}$ and $\lambda_i$ are the eigenvalues of $\mathrm{d}F(\overline{W})$. Hence

$$\sum_{i=1}^{N} k^i(x - \lambda_i t) \mathrm{d}G(\overline{W})\, r_i = 0 \qquad \forall\, (x,t). \tag{10}$$

Let $K_1, \dots, K_N : \mathbb{R} \to \mathbb{R}^N$ be $N$ differentiable functions and $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ be such that $\lambda_i \neq \lambda_j$ if $i \neq j$. If $\sum_{i=1}^{N} K_i(x - \lambda_i t) = \sum_{i=1}^{N} K_i(0)$, then $K_i(x - \lambda_i t) = K_i(0)$ for any $(x,t)$ and for any $i$.

**Proof.** Without loss of generality we can assume $K_i(0) = 0$ for any $i$ and we prove the Lemma by induction on $N$. The case $N = 1$ is trivial. Assume that the thesis hold for $N - 1$. By assumption,

$$K_N(s) = -\sum_{i=1}^{N-1} K_i(s + (\lambda_N - \lambda_i)t) \qquad \forall\,(s,t). \tag{11}$$

Differentiating the equality $\sum_{i=1}^{N} K_i(x - \lambda_i t) = 0$ with respect to $x$ and $t$, we get

$$\sum_{i=1}^{N} K_i'(x - \lambda_i t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \lambda_i K_i'(x - \lambda_i t) = 0.$$

From these relations it follows

$$K_N'(s) = -\sum_{i=1}^{N-1} \frac{\lambda_i - \lambda_1}{\lambda_N - \lambda_1} K_i'(s + (\lambda_N - \lambda_i)t).$$

Integrating

$$K_N(s) = -\sum_{i=1}^{N-1} \frac{\lambda_i - \lambda_1}{\lambda_N - \lambda_1} K_i(s + (\lambda_N - \lambda_i)t).$$

Subtracting this relation to (11), we get

$$\sum_{i=1}^{N-1} \frac{\lambda_N - \lambda_i}{\lambda_N - \lambda_1}\, K_i(s + (\lambda_N - \lambda_i)t) = 0.$$

By the inductive assumption, $\frac{\lambda_N - \lambda_i}{\lambda_N - \lambda_1} K_i$ are constant for any $i = 1, \dots, N-1$, hence all of the functions $K_n$ are constant. This concludes the proof. □

Applying Lemma 2 to (10), we deduce

$$k^i(x - \lambda_i t) \mathrm{d}G(\overline{W})\, r_i = k^i(0) \mathrm{d}G(\overline{W})\, r_i \qquad \forall\, i = 1, \dots, N.$$

In particular, the vectors $(k^i(x) - k^i(0))r_i$ for $i = 1, \dots, N$ belongs to $\ker \mathrm{d}G(\overline{W})$ and are eigenvectors of $\mathrm{d}F(\overline{W})$. Since, by assumption, the solution $w$, given in (9), is nonconstant, $k^i(x) - k^i(0) \neq 0$ for some $i$ and for some $x$, hence the Shizuta–Kawashima condition is not satisfied.

In the one–dimensional strictly hyperbolic case, the linearized system (8) satisfies the Shizuta–Kawashima condition if and only if all of its maxwellians are constant.

At the nonlinear level, the equivalence does not hold. As an example consider the Jin–Xin system (6) with the additional assumption on $h$:

$$\begin{aligned} &\exists\, \bar u \in \mathbb{R}^n \setminus \{0\} \quad \text{such that} \quad \mathrm{d}h(0)\, \bar u = a\, \bar u, \\ &a^2 I - \mathrm{d}h(u)^2 > 0 \qquad \forall\, u \neq 0. \end{aligned} \tag{12}$$

Under these assumptions, the Shizuta–Kawashima condition is not satisfied, since $(\bar u, \mathrm{d}h(0)\bar u) \in \ker \mathrm{d}G(0, h(0))$ is an eigenvector of $\mathrm{d}F(0, h(0))$. Nevertheless, there exists no nonconstant maxwellian, since the relation

$$\left[a^2 I - \mathrm{d}h(U)^2\right] U_x = 0 \tag{13}$$

implies $U_x = 0$ or $U = 0$, hence $U$ constant.

Nevertheless a link beetween the Shizuta–Kawashima condition still holds also in the nonlinear case. As an example, if $W$ is a maxwellian for (5) and $W$ has the form $W(x,t) = W(x - \lambda t)$ for some $\lambda \in \mathbb{R}$, then

$$\mathrm{d}\,G(W)W_x = 0,\ \mathrm{d}F(W)W_x = \lambda W_x.$$

Hence $W_x$ is in the kernel of $\mathrm{d}G(W)$ and is an eigenvector (with eigenvalue $\lambda$) of $DF(W)$. Therefore the Shizuta–Kawashima condition is not satisfied at any $W$ such that $W_x \neq 0$.

Similar conclusion hold if the maxwellian $W$ is of the form $W(x,t) = H(x/t)$ (as in the rarefaction case) for some smooth function $H$. Then

$$\mathrm{d}F(W)H' = -\frac{x}{t}\,H', \qquad \mathrm{d}G(W)H' = 0, \tag{14}$$

hence the Shizuta–Kawashima condition is not satisfied.

As an example of a physical hyperbolic relaxation systems possessing equilibrium solutions, let us consider the Kerr–Debye model, arising in the analysis of nonlinear optical phenomena (see [15, 8, 9]):

$$\begin{cases} d_t + b_x = 0 \\ b_t + \big(d(1+\chi)^{-1}\big)_x = 0 \\ \chi_t = d^2(1+\chi)^{-2} - \chi. \end{cases} \tag{15}$$

In this case $u = (d,b) \in \mathbb{R}^2$ and $v = \chi \in \mathbb{R}$. Let us introduce the variables $(\alpha, \beta, \gamma)$

$$\alpha = \frac{d}{1+\chi}, \qquad \beta = b, \qquad \gamma = \frac{1}{2}\left(\chi - \frac{d^2}{(1+\chi)^2}\right),$$

so that $(d, b, \chi) = (\alpha(1 + \alpha^2 + 2\gamma), \beta, \alpha^2 + 2\gamma)$. The system (15) takes the form

$$\begin{cases} (1+3\alpha^2)\alpha_t + 2\alpha\gamma_t + \beta_x = 0 \\ \beta_t + \left(1 - \dfrac{2\gamma}{1+\alpha^2+2\gamma}\right)\alpha_x = 0 \\ 2\alpha\alpha_t + 2\gamma_t = -2\gamma \end{cases}$$

Looking for solution such that $\gamma = 0$, we get

$$(1+3\alpha^2)\alpha_t + \beta_x = 0, \qquad \beta_t + \alpha_x = 0 \qquad \alpha\alpha_t = 0$$

Assuming $\alpha$ non constant, we get from the third equation $\alpha = \alpha(x)$ and $\beta = \beta(t)$. Hence from the second equation, we deduce $\alpha'(x) = -\beta'(t) = \lambda \in \mathbb{R}$. Therefore, equilibrium solutions for (15) have the form

$$\alpha = \lambda x + \alpha_0, \qquad \beta = -\lambda t + \beta_0, \qquad \gamma = 0.$$

with $\alpha_0, \beta_0, \lambda \in \mathbb{R}$. In term of the original variables, these solutions are

$$\begin{cases} d(x,t) = (\lambda\,x + \alpha_0)(1 + (\lambda\,x + \alpha_0)^2), \\ b(x,t) = -\lambda t + \beta_0, \\ \chi(x,t) = (\lambda\,x + \alpha_0)^2. \end{cases}$$

being constant if and only if $\lambda = 0$. Notice that for this systems, condition [SK] does not hold at $b = 0$, see [16].

## 3. Dissipative entropies and $L^2$ estimates

Once an equilibrium solution $W$ has been chosen, the main problem is to determine its stability (in particular on constant steady states) under suitable assumptions on the system (5). In this Section, we introduce the concept of dissipative entropy and we show how the presence of such a functional permits to determine some control on the $L^2$-norm of perturbation of any maxwellian. This is by no means sufficient for proving a complete stability result (see next Section for some stability results and discussions on the argument).

A convex entropy for (5) is a real-valued function $\mathcal{E} = \mathcal{E}(w)$ such that

$$\mathrm{d}^2\mathcal{E} \geq 0 \qquad \text{and} \qquad \mathrm{d}\mathcal{E}\,\mathrm{d}F_i = \mathrm{d}\mathcal{F}_i \qquad \forall\, i = 1, \ldots, d.$$

for some real-valued differentiable functions $\mathcal{F}_i$.

Locally, a convex function $\mathcal{E}$ defines an entropy if and if $\mathrm{d}(\mathrm{d}\mathcal{E}\mathrm{d}F_i)$ is symmetric for any $i$. Since $\mathrm{d}\mathcal{E}\mathrm{d}^2F_i$ is always symmetric, $\mathcal{E}$ is an entropy if and only if $\mathrm{d}^2\mathcal{E}\mathrm{d}F_i$ is symmetric for any $i$.

If $F_i(w) = dH_i(w)$ for some $H_i \in C^2(\mathbb{R}^N, \mathbb{R})$, so that $\mathrm{d}F_i(w) = \mathrm{d}^2H_i(w)$ is a symmetric $N \times N$–matrix, an entropy of the system (5) is given by $\mathcal{E}(w) := \frac{1}{2}|w|^2$. Check of this assertion is immediate, since $\mathrm{d}^2\mathcal{E}(w) = I$ for any $w$.

An entropy for the one-dimensional Kerr–Debye system (15) is

$$\mathcal{E}(d, b, \chi) = \frac{1}{2}\left(\frac{d^2}{1+\chi} + b^2 + \frac{1}{2}\chi^2\right).$$

Indeed

$$\mathrm{d}\mathcal{E}\,\mathrm{d}F = \left(\frac{b}{1+\chi}, \frac{d}{1+\chi}, -\frac{db}{(1+\chi)^2}\right) = d\mathcal{F}$$

where $\mathcal{F}(d, b, \chi) = db(1+\chi)^{-1}$.

When a system is endowed with an entropy, an additional equation for the entropy evolution can be written: fixed $\overline{w} \in \mathcal{V} \subset \mathbb{R}^N$, taking the scalar product of (5) against $\mathrm{d}\mathcal{E}(w) - \mathrm{d}\mathcal{E}(\overline{w})$:

$$\frac{\partial}{\partial t}\big(\mathcal{E}(w) - \langle \mathrm{d}\mathcal{E}(\overline{w}), w\rangle\big) + \mathrm{div}_x\left(\mathcal{F}(w) - \mathrm{d}\mathcal{E}(\overline{w})\,F(w)\right)$$
$$= \langle \mathrm{d}\mathcal{E}(w) - \mathrm{d}\mathcal{E}(\overline{w}), G(w) - G(\overline{w})\rangle.$$

The integral of $\mathcal{E}(w) - \mathrm{d}\mathcal{E}(\overline{w})w$ is decreasing if the term on the righthand side is negative. This property is encoded in the following definition.

An entropy $\mathcal{E}$ for the system (5) is dissipative at $\overline{w} \in \mathcal{V}$ if, for any $w$ in a neighborhood of $\overline{w}$,

$$\langle \mathrm{d}\mathcal{E}(w) - \mathrm{d}\mathcal{E}(\overline{w}), G(w) - G(\overline{w})\rangle \leq 0. \tag{16}$$

An entropy $\mathcal{E}$ is dissipative if it is dissipative at $\overline{w}$ for any $\overline{w} \in \mathcal{V}$.

According to [16], if the system has a dissipative entropy, then, possibly using of a simple change of variables, we can choose the entropy, such that , for a single equilibrium value $\overline{w}$, $\mathrm{d}\mathcal{E}(\overline{w}) = 0$, i.e.: it is a quadratic function, so that the dissipation condition at tht point just reads as $\langle \mathrm{d}\mathcal{E}(w), G(w)\rangle \leq 0$. In the following, all the entropies will be chosen to be quadratic in one point.

If $\mathrm{d}F$ is symmetric, an entropy for (5) is $\mathcal{E} = \frac{1}{2}|w|^2$. By definition this entropy is dissipative if and only if $\langle G(w), w - \overline{w}\rangle \leq 0$ for any $w$ in a neighborhood of $\overline{w}$.

Let $\overline{w}_1, \overline{w}_2 \in \mathcal{V}$, then the dissipation condition written for $\overline{w} = \overline{w}_2$ and $w = \overline{w}_1 + th$ for some $h \in \mathbb{R}^N$, $t \in \mathbb{R}$, becomes $\langle \overline{w}_1 - \overline{w}_2 + th, G(\overline{w}_1 + th)\rangle \leq 0$ for any $t \in \mathbb{R}$ and $h \in \mathbb{R}^N$. Hence we deduce that

$$\langle \overline{w}_1 - \overline{w}_2, \mathrm{d}G(\overline{w}_1)\, h\rangle \leq 0 \qquad\qquad \forall\, h \in \mathbb{R}^N.$$

Therefore $\langle \overline{w}_1 - \overline{w}_2, \mathrm{d}G(\overline{w}_1)\rangle = 0$, that is $\overline{w}_1 - \overline{w}_2 \in \ker(\mathrm{d}G(\overline{w}_1)^T)$. In other words, $\overline{w}_2 \in \overline{w}_1 + \ker(\mathrm{d}G(\overline{w}_1)^T)$. Hence we can conclude that, in the symmetric case, $\frac{1}{2}\,|u|^2$ is a dissipative entropy if and only if the equilibrium manifold $\mathcal{V}$ is a linear manifold. This remark has been developed in [16], where it was shown that, taking as a new unknown the entropy variable $W = \mathrm{d}\mathcal{E}(w)$, the new system is symmetric and the equilibrium set reduces to a linear manifold.

Next we want to use dissipative entropy to get *a priori* $L^2-$estimate for perturbations of some fixed maxwellian of the system (5). Let $\tilde{w}$ and $W$ be smooth solutions of the hyperbolic system (5). We aim to determine $L^2$ estimates for the distance between the two solution (at the end $W$ will be a maxwellian), i.e. we want to estimate $|w|_{L^2}$ where the perturbation $w := \tilde{w} - W$. solves

$$w_t + \mathrm{div}_x \big[F(W+w) - F(W)\big] = G(W+w) - G(W) \tag{17}$$

Given a dissipative entropy $\mathcal{E} = \mathcal{E}(w)$ and a reference solution $W$, we call modulated entropy $J$ at state $W$

$$J(w;W) := \mathcal{E}(W+w) - \mathcal{E}(W) - \langle \mathrm{d}\mathcal{E}(W), w\rangle. \tag{18}$$

Since $J(w;W) \approx \langle w, \mathrm{d}^2E(W)w\rangle$ as $w \to 0$, $J$ will be useful in determining the $L^2$ estimates. If $\mathcal{E}(w) = \frac{1}{2}\langle E_0 w, w\rangle$, then $J(w;W) = \mathcal{E}(w)$.

By direct computation, using the symmetry of $\mathrm{d}^2\mathcal{E}$, the equation (17), and the relations

$$\begin{aligned}
\frac{\partial J}{\partial t} &= -\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), \mathrm{div}_x \big(F(W+w) - F(W)\big)\rangle \\
&\quad +\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), G(W+w) - G(W)\rangle \\
&\quad +\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W) - \mathrm{d}^2\mathcal{E}(W)w, W_t\rangle \\
\frac{\partial}{\partial x}\Big(&\mathcal{F}_i(W+w) - \mathcal{F}_i(W) - \langle \mathrm{d}\mathcal{E}(W), F_i(W+w) - F_i(W)\rangle\Big) \\
&= \langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), \mathrm{d}F_i(W+w)_{x_i}\rangle \\
&\quad -\langle \mathrm{d}^2\mathcal{E}(W)W_{x_i}, F_i(W+w) - F_i(W)\rangle,
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{\partial J}{\partial t} + \mathrm{div}_x K = &-\sum_{i=1}^{d} \langle \mathrm{d}^2\mathcal{E}(W)W_{x_i}, F_i(W+w) - F_i(W)\rangle \\
&+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), \mathrm{div}_x F(W)\rangle \\
&+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), G(W+w) - G(W)\rangle \\
&+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W) - \mathrm{d}^2\mathcal{E}(W)w, W_t\rangle
\end{aligned}$$

where $K = (K_1, \dots, K_d)$ and

$$K_i = \mathcal{F}_i(W+w) - \mathcal{F}_i(W) - \langle \mathrm{d}\mathcal{E}(W), F_i(W+w) - F_i(W)\rangle$$

Since $W$ is a solution itself, the previous relation can be rewritten as

$$\frac{\partial J}{\partial t} + \mathrm{div}_x K = -\sum_{i=1}^{d} \langle \mathrm{d}^2\mathcal{E}(W)W_{x_i}, F_i(W+w) - F_i(W)\rangle$$

$$+\langle \mathrm{d}^2\mathcal{E}(W)w, \mathrm{div}_x\, F(W)\rangle$$
$$+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W) - \mathrm{d}^2\mathcal{E}(W)w, G(W)\rangle$$
$$+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), G(W+w) - G(W)\rangle$$

Finally, since $\mathrm{d}^2\mathcal{E}$ and $\mathrm{d}^2\mathcal{E}\mathrm{d}F_i$ are symmetric,

$$\langle \mathrm{d}^2\mathcal{E}\, w, F_i(W)_{x_i}\rangle = \langle \mathrm{d}^2\mathcal{E}\, w, \mathrm{d}F_i(W)W_{x_i}\rangle = \langle \mathrm{d}^2\mathcal{E}\, W_{x_i}, \mathrm{d}F_i(W)w\rangle$$

we obtain

$$\frac{\partial J}{\partial t} + \mathrm{div}_x \quad K = -\sum_{i=1}^{d}\langle \mathrm{d}^2\mathcal{E}(W)W_{x_i}, F_i(W+w) - F_i(W) - \mathrm{d}F_i(W)w\rangle$$
$$+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W) - \mathrm{d}^2\mathcal{E}(W)w, G(W)\rangle$$
$$+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), G(W+w) - G(W)\rangle \tag{19}$$

Under additional assumption on the system (5), the relation (19) simplifies:
– either for constant reference solutions $W$ or in the semilinear case, i.e. $F_i(w) = A_i\, w$ for some constant matrices $A_i$, the first term at the righthand side disappears;
– if the entropy $\mathcal{E}$ is a quadratic form, i.e. $\mathcal{E}(w) := \frac{1}{2}\langle w, E_0 w\rangle$ for some symmetric $E_0 > 0$, the second term is zero.

If $W$ is an equilibrium solution, i.e. if $G(W) = 0$, relation (19) becomes

$$\frac{\partial J}{\partial t} + \mathrm{div}_x \quad K = -\sum_{i=1}^{d}\langle \mathrm{d}^2\mathcal{E}(W)W_{x_i}, F_i(W+w) - F_i(W) - \mathrm{d}F_i(W)w\rangle$$
$$+\langle \mathrm{d}\mathcal{E}(W+w) - \mathrm{d}\mathcal{E}(W), G(W+w) - G(W)\rangle \tag{20}$$

As soon as the perturbation $w$ is small, the previous equation can be written in a more significant form as

$$\frac{\partial J}{\partial t} + \mathrm{div}_x\, K = -\mathcal{B}(w,w) + o(|w|^2) \tag{21}$$

where $\mathcal{B}$ denotes the bilinear form

$$\mathcal{B}(W; w, w) := \sum_{i=1}^{d}\langle \mathrm{d}^2\mathcal{E}(W)W_{x_i}, \mathrm{d}^2F_i(W)w\, w\rangle - \langle \mathrm{d}^2\mathcal{E}(W)w, \mathrm{d}G(W)w\rangle \tag{22}$$

and

$$\mathrm{d}^2F_i(W)w\, w = \sum_{j,k=1}^{N}\frac{\partial^2 F_i(W)}{\partial w_j \partial w_k} w_j\, w_k \qquad\qquad i = 1, \dots, d.$$

Therefore we propose the following stability condition.

**Stability condition** `[St]`. *A necessary condition for the modulated entropy to not increase is that $\mathcal{B}$, defined in (22), is positive semidefinite.*

If $W$ is constant, this follows from the request of $\mathcal{E}$ to be a dissipative entropy. In the nonconstant case, the necessary condition is perturbed by the presence of the first term in the definition of $\mathcal{B}$.

If the entropy $\mathcal{E}$ is quadratic, i.e. $\mathcal{E}(w) := \frac{1}{2}\langle w, E_0 w\rangle$ for some symmetric $E_0 > 0$, the above condition can be rewritten in a simpler way. Indeed, from symmetry of $E_0\mathrm{d}F_i$ for any $i$, it follows

$$\langle \mathrm{d}^2\mathcal{E}(W)W_{x_i}, \mathrm{d}^2F_i(W)w\, w\rangle = \langle \mathrm{d}^2\mathcal{E}(W)w, \partial_x(\mathrm{d}F_i(W))w\rangle.$$

Hence, $\mathcal{B}$ becomes

$$\mathcal{B}(W;w,w) = \langle w, \mathrm{d}^2\mathcal{E}(W)\big(\mathrm{div}_x\,(\mathrm{d}F(W)) - \mathrm{d}G(W)\big)w\rangle$$

Since $E_0$ is positive definite, the necessary condition takes the form

$$\mathrm{div}_x\,(\mathrm{d}F(W)) - \mathrm{d}G(W) \geq 0 \tag{23}$$

for any $(x,t)$. Since *a priori* there is no control on the sign of components of $\nabla W$, this condition should be intepreted as a request of degeneracy of $\mathrm{d}^2F$ along the kernel of $\mathrm{d}G$. Poorly speaking, the flux $F$ has to be linear along the equilibrium manifold defined by $G$.

Let $x$ be one-dimensional, $F$ such that $\mathrm{d}F$ is symmetric and $G$ of the form described in [H] with $q(u,v) = -b\,v$ for some $p\times p$ positive definite matrix $b$. In this case $\mathcal{E}(w) = \frac{1}{2}|w|^2$ is a dissipative entropy for (5). The stability condition (23) for a maxwellian $W = (U,0)$ becomes

$$\begin{pmatrix} \partial_x(\mathrm{d}_u f(U,0)) & \partial_x(\mathrm{d}_v f(U,0)) \\ \partial_x(\mathrm{d}_u g(U,0)) & \partial_x(\mathrm{d}_v g(U,0)) + b \end{pmatrix} \geq 0$$

In particular, this is satisfied if

$$\mathrm{d}^2_{uu} f(u,0) = \mathrm{d}^2_{uv} f(u,0) = 0 \qquad\qquad \forall\, u.$$

(by simmetry of $\mathrm{d}F$, $\mathrm{d}_u g = (\mathrm{d}_v f)^t$) and $U_x$ is assumed to be sufficiently small. A specific choice for $f$ is $f(u,v) = A_0\,u + \tilde{f}(v)$ where $A_0$ is an $n\times n$ constant matrix and $\tilde{f}$ a smooth function.

If the entropy $\mathcal{E}$ and the flux functions $F_i$ are quadratic, and the reaction term $G$ is linear the $o(|w|^2)$ term in (21) is zero, hence it is possible to prove a result on $L^2-$stability under structural additional assumptions. The hypotheses are strong and the systems fitting in this framework should be considered only as toy models. Nevertheless, we consider interesting to build up some oversimplified model, where rigourous results can be proved.

For simplicity, we consider the one-dimensional case, i.e. $x \in \mathbb{R}$.

Assume [H]. Let $F$ be a quadratic function, $G$ of the form

$$G(u,v) = -Bw := -\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $b > 0$ is a costant $p\times p$ matrix and let $\mathcal{E}(w) := \frac{1}{2}\langle w, E_0 w\rangle$ for some symmetric constant $N\times N$ matrix $E_0 > 0$ of the form

$$E_0 = \begin{pmatrix} E_{11} & 0_{n\times p} \\ 0_{p\times n} & E_{22} \end{pmatrix}.$$

Let $W = (U,0)$ be an equilibrium solution of (3) such that

$$\mathrm{d}^2_{uu} f(U,0)\,U_x = \mathrm{d}^2_{uv} f(U,0)\,U_x = \mathrm{d}^2_{uu}(U,0)\,U_x = 0$$

for any $(x,t)$.

Then if $U_x$ is small, any solution of (17) with initial data $w_0 \in L^2(\mathbb{R})$ is such that

$$|w|^2_{L^2}(t) + \int_0^t |v|^2_{L^2}(\tau)\,d\tau \leq |w_0|^2_{L^2}$$

for any $t > 0$.

**Proof.** Since $o(|w|^2) = 0$, integrating in $\mathbb{R} \times [0,t]$ equation (21), we get

$$\int_{\mathbb{R}} \langle E_0 w, w\rangle(x,t)\,dx + 2\int_0^t \int_{\mathbb{R}} \mathcal{B}(w,w)(x,\tau)\,dx\,d\tau = \int_{\mathbb{R}} \langle E_0 w_0, w_0\rangle(x,t)\,dx$$

where $\mathcal{B}$ is the quadratic with coefficient's matrix given by

$$E_0\big(\partial_x(\mathrm{d}F(W)) + BW\big) = E_0\begin{pmatrix} 0 & 0 \\ 0 & E_{22}\big(b + \partial_x(\mathrm{d}_v g(W))\big)\end{pmatrix}.$$

Since $b > 0$, this matrix is positive definite if $U_x$ is small enough. Hence the conclusion holds. □

If $F$ is such that $\mathrm{d}F$ is symmetric, then $E_0 = I$. Moreover, assuming $F = (f,g)$ such that:

$$f(u,v) = A_0\,u + \tilde f(v),\;\; g(u,v) = \mathrm{d}_v \tilde f(v) u + \tilde g(v), \tag{24}$$

for a symmetric matrix $A_0$ and some smooth functions $\tilde f(v)$ and $\tilde g(v)$, such that $\mathrm{d}_v\tilde f(v)$, $\mathrm{d}_{vv}\tilde f(v)$, and $\mathrm{d}_v\tilde g(v)$ are symmetric, then the above assumptions are satisfied. In Section 5, we start back from this kind of example and we show that a complete stability result can be proved.

Next, we aim to determine necessary and sufficient conditions for the existence of dissipative entropies for a given system of the form (5). In the $2\times 2$ case, i.e. $n = p = 1$, a more detailed description of the problem can be given, see Section 4. We consider the one–dimensional case, i.e. we deal with

$$W_t + F(W)_x = G(W).$$

where $F$ and $G$ satsfies assumption [H]. Thus, the system can be rewritten as

$$\begin{cases} u_t + f(u,v)_x = 0 \\ v_t + g(u,v)_x = q(u,v)\end{cases} \tag{25}$$

With $h$ as in [H], the relaxed system of (25) is

$$u_t + f_*(u)_x = 0 \qquad\qquad f_*(u) := f(u,h(u)) \tag{26}$$

Moreover set $g_*(u) := g(u,h(u))$.

Whenever the system (25) has an entropy $\mathcal{E}$, it is natural to ask if the restriction of this entropy, $\eta(u) = \mathcal{E}(u,h(u))$ is an entropy for the reduced system (26). At the same time it is interesting to understand when an entropy for the reduced system can be extended to an entropy of the original system. Notice that interesting results in this direction can be also found in [5]. Assume [H] and let $\mathcal{E}$ be an entropy for system (25). Then

$$(\eta(u),\theta(u)) := (\mathcal{E}(u,h(u)), \mathcal{F}(u,h(u)))$$

is an entropy/entropy flux for the relaxed system (26) if and only if on the equilibrium manifold $\mathcal{V}$ there holds

$$\mathrm{d}_v\mathcal{E}\Big(\mathrm{d}h\,\mathrm{d}f_* - \mathrm{d}g_*\Big) = 0. \tag{27}$$

**Proof.** By definition of entropy, the condition $\mathrm{d}\eta\,\mathrm{d}f_* = \mathrm{d}\theta$ has to be satisfied. Since

$$\mathrm{d}\eta = \mathrm{d}_u\mathcal{E} + \mathrm{d}_v\mathcal{E}\,\mathrm{d}h, \qquad \mathrm{d}f_* = \mathrm{d}_u f + \mathrm{d}_v f\,\mathrm{d}h, \qquad \mathrm{d}\eta = \mathrm{d}_u\mathcal{F} + \mathrm{d}_v\mathcal{F}\,\mathrm{d}h,$$

hence the following condition has to hold

$$\mathrm{d}_v\mathcal{E}\,\mathrm{d}h\,\mathrm{d}_v f\,\mathrm{d}h + \mathrm{d}_v\mathcal{E}\,\mathrm{d}h\,\mathrm{d}_u f + [\mathrm{d}_u\mathcal{E}\,\mathrm{d}_v f - \mathrm{d}_v\mathcal{F}]\,\mathrm{d}h + \mathrm{d}_u\mathcal{E}\,\mathrm{d}_u f - \mathrm{d}_u\mathcal{F} = 0. \tag{28}$$

Condition $\mathrm{d}\mathcal{E}\,\mathrm{d}F = \mathrm{d}\mathcal{F}$ can be rewritten as

$$\mathrm{d}_u\mathcal{F} = \mathrm{d}_u\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}_v\mathcal{E}\,\mathrm{d}_u g \qquad \mathrm{d}_v\mathcal{F} = \mathrm{d}_u\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}_v\mathcal{E}\,\mathrm{d}_v g$$

thus (28) becomes (27), since $\mathrm{d}g_* = \mathrm{d}_u g + \mathrm{d}_v g\,\mathrm{d}h$. □

Without loss of generality, we may assume that $(0,0)$ is an equilibrium point: $\bar{u} = 0$, $h(0) = 0$, and that we can choose the quadratic entropy such that $\mathrm{d}\mathcal{E}(0,0) = 0$. Assume [H] and let $\mathcal{E}$ be a quadratic dissipative entropy with respect to the point $(0,0)$ for the system (25). Then, for any $u$,

$$\mathrm{d}_v\mathcal{E}(u,h(u)) = 0 \qquad \text{and} \qquad \mathrm{d}_v^2\mathcal{E}(u,h(u)) \geq 0. \tag{29}$$

In particular, $(\eta(u),\theta(u)) := (\mathcal{E}(u,h(u)),\mathcal{F}(u,h(u)))$, is an entropy/entropy flux for (26).

**Proof.** Since $\mathrm{d}\mathcal{E} = (\mathrm{d}_u\mathcal{E},\mathrm{d}_v\mathcal{E})$ and $G = (0,q)$, the dissipation condition yields $\mathrm{d}_v\mathcal{E}\,q(u,v) \leq 0$ for any $(u,v)$. Since $q(u,h(u)) = 0$ and, from (4), $\mathrm{d}_v q(u,h(u)) < 0$, then $q(u,h(u)+\varepsilon) < 0 < q(u,h(u)-\varepsilon)$ for $\varepsilon$ small enough. Therefore $\mathrm{d}_v\mathcal{E}(u,h(u)-\varepsilon) \leq 0 \leq \mathrm{d}_v\mathcal{E}(u,h(u)+\varepsilon)$ for $\varepsilon$ sufficiently small. For $\varepsilon \to 0$ we get the conclusion. □

Since for a system of the form (25) dissipative of the entropy is equivalent to

$$\big[\mathrm{d}_v\mathcal{E}(u,v) - \mathrm{d}_v\mathcal{E}(u,h(u))\big]q(u,v) \leq 0$$

if there holds

$$\mathrm{d}_v\mathcal{E}(u,h(u)) = 0 \quad \forall\, u \qquad \text{and} \qquad \mathrm{d}_{vv}^2\mathcal{E}(u,v) \geq 0 \quad \forall\, u,v,$$

then $\mathcal{E}$ is dissipative. In any case, it is often preferable to have conditions to be checked only on the equilibrium manifold $\mathcal{V}$. The following Proposition is obvious since $\mathcal{E} \in C^2$.

Assume [H] and let $\mathcal{E} \in C^2$ be a quadratic entropy for system (25). If, for some fixed $u$,

$$\mathrm{d}_v\mathcal{E}(u,h(u)) = 0 \qquad \text{e} \qquad \mathrm{d}_{vv}^2\mathcal{E}(u,h(u)) > 0, \tag{30}$$

then the entropy $\mathcal{E}$ is dissipative at $(u,h(u))$.

Note that *an entropy for (25) can be dissipative without being strictly convex.* The strict convexity is needed transversally to the equilibrium manifold $\mathcal{V}$.

A function $\mathcal{E}$ is an entropy if $\mathrm{d}\mathcal{E}\,\mathrm{d}F$ is (locally) a gradient. Hence necessary and sufficient condition is that the matrix $\mathrm{d}(\mathrm{d}\mathcal{E}\,\mathrm{d}F)$ is symmetric. Since $\mathrm{d}\mathcal{E}\,\mathrm{d}^2F$ is always symmetric if $F \in C^2$, the condition reduce to the request: $\mathrm{d}^2\mathcal{E}\,\mathrm{d}F$ is symmetric. Since

$$\mathrm{d}^2\mathcal{E}\,\mathrm{d}F = \begin{pmatrix} \mathrm{d}_{uu}^2\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}_{vu}^2\mathcal{E}\,\mathrm{d}_u g & \mathrm{d}_{uu}^2\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}_{vu}^2\mathcal{E}\,\mathrm{d}_v g \\ \\ \mathrm{d}_{uv}^2\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}_{vv}^2\mathcal{E}\,\mathrm{d}_u g & \mathrm{d}_{uv}^2\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}_{vv}^2\mathcal{E}\,\mathrm{d}_v g \end{pmatrix}$$

symmetry holds if and only if the matrices $\mathrm{d}_{uu}^2\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}_{vu}^2\mathcal{E}\,\mathrm{d}_u g$ and $\mathrm{d}_{uv}^2\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}_{vv}^2\mathcal{E}\,\mathrm{d}_v g$ are symmetric and

$$\big(\mathrm{d}_{uu}^2\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}_{vu}^2\mathcal{E}\,\mathrm{d}_v g\big)^t = \mathrm{d}_{uv}^2\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}_{vv}^2\mathcal{E}\,\mathrm{d}_u g.$$

Hence we get

$$(\mathrm{d}_v f)^t\mathrm{d}_{uu}^2\mathcal{E} - \mathrm{d}_{uv}^2\mathcal{E}\,\mathrm{d}_u f + (\mathrm{d}_v g)^t\mathrm{d}_{uv}^2\mathcal{E} - \mathrm{d}_{vv}^2\mathcal{E}\,\mathrm{d}_u g = 0. \tag{31}$$

The "initial" condition $\mathcal{E}$ are

$$\mathcal{E}(u,h(u)) = \eta(u), \qquad \mathrm{d}_v\mathcal{E}(u,h(u)) = 0,$$

so that, on the equilibrium manifold $\mathcal{V}$,

$$\mathrm{d}_u\mathcal{E} + \mathrm{d}_v\mathcal{E}\mathrm{d}h = \mathrm{d}\eta \qquad \mathrm{d}^2_{uv}\mathcal{E} + \mathrm{d}^2_{vv}\mathcal{E}\,\mathrm{d}h = 0,$$

$$\mathrm{d}^2_{uu}\mathcal{E} + (\mathrm{d}h)^t\mathrm{d}^2_{uv}\mathcal{E} + \mathrm{d}^2_{vu}\mathcal{E}\,\mathrm{d}h + (\mathrm{d}h)^t\mathrm{d}^2_{vv}\mathcal{E}\,\mathrm{d}h + \mathrm{d}_v\mathcal{E}\mathrm{d}^2_{vv}\,h = \mathrm{d}^2\eta.$$

Using these relations to obtain an equality only in term $\mathrm{d}^2_{vv}\mathcal{E}$, we obtain (thanks to the simmetry of $\mathrm{d}^2_{vv}\mathcal{E}$),

$$\mathrm{d}_v f^t\mathrm{d}^2\eta = (\mathrm{d}_v g - \mathrm{d}h\,\mathrm{d}_v f)^t\mathrm{d}^2_{vv}\mathcal{E}\,\mathrm{d}h' + \mathrm{d}^2_{vv}\mathcal{E}\,(\mathrm{d}_u g - \mathrm{d}\,h\,\mathrm{d}_u f).$$

Let $A^* := A - A^T$ be the antisymmetric part of the matrix $A$, this can be rewritten as

$$\mathrm{d}_v f^t\mathrm{d}^2\eta = \left[\mathrm{d}^2_{vv}\mathcal{E}(\mathrm{d}h\,\mathrm{d}_v f - \mathrm{d}_v g)\right]^*\mathrm{d}h - \mathrm{d}^2_{vv}\mathcal{E}\big(\mathrm{d}h\mathrm{d}f_* - \mathrm{d}g_*\big). \tag{32}$$

This relation has to be satisfied together with

$$\begin{aligned} &\mathrm{d}^2_{uu}\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}^2_{vu}\mathcal{E}\,\mathrm{d}_u g = \Big(\mathrm{d}^2_{uu}\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}^2_{vu}\mathcal{E}\,\mathrm{d}_u g\Big)^t, \\ &\mathrm{d}^2_{uv}\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}^2_{vv}\mathcal{E}\,\mathrm{d}_v g = \Big(\mathrm{d}^2_{uv}\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}^2_{vv}\mathcal{E}\,\mathrm{d}_v g\Big)^t. \end{aligned} \tag{33}$$

Writing everything in term of $\mathrm{d}^2_{vv}\mathcal{E}$, we deduce from (33), that $\left[\mathrm{d}^2_{vv}\mathcal{E}(\mathrm{d}h\,\mathrm{d}_v f - \mathrm{d}_v g)\right]^*\mathrm{d}h = 0$. So, we can prove the following result. Let $\mathcal{E}$ a dissipative entropy for system (25). Setting $f_* = f(u, h(u))$, $g_* = g(u, h(u))$, we have, on the equilibrium manifold $v = h(u)$, the relation

$$\mathrm{d}_v f^t\mathrm{d}^2\eta = \mathrm{d}^2_{vv}\mathcal{E}\big(-\mathrm{d}h\,\mathrm{d}f_* + \mathrm{d}g_*\big). \tag{34}$$

Moreover the matrices $\mathrm{d}^2_{uu}\mathcal{E}\,\mathrm{d}_u f + \mathrm{d}^2_{vu}\mathcal{E}\,\mathrm{d}_u g$ and $\mathrm{d}^2_{uv}\mathcal{E}\,\mathrm{d}_v f + \mathrm{d}^2_{vv}\mathcal{E}\,\mathrm{d}_v g$ are symmetric. Applying $\mathrm{d}_v f$ at the right-hand side of (34), we obtain the more symmetric form

$$\langle \mathrm{d}_v f, \mathrm{d}^2\eta\,\mathrm{d}_v f\rangle = \mathrm{d}^2_{vv}\mathcal{E}\big(-\mathrm{d}h\,\mathrm{d}f_* + \mathrm{d}g_*\big)\,\mathrm{d}_v f. \tag{35}$$

Formulas (34)–(35) give an interesting and non trivial connection between the second derivative of the entropy $\eta$ of the reduced system, with the second derivative with respect to $v$ of the entropy $\mathcal{E}$ of the complete system.

*Relation to the Chapman–Enskog expansion.* Let $n = p$ and, for simplicity, $q(u, v) = h(u) - v$. The Chapman–Enskog expansion consists in expanding $v$ as $v = h(u) - \varepsilon d(u)\,u_x + o(\varepsilon)$ where $d = d(u)$ is an $n \times n$ matrix to be determined. Inserting in the second equation of (25) and collecting $O(\varepsilon)-$terms, we obtain $d(u) = -\mathrm{d}h\,\mathrm{d}f_* + \mathrm{d}g_*$.

Hence for $\epsilon$ small, a (viscous) system formally approximating (25) is

$$u_t + f_*(u)_x = \varepsilon(D(u)\,u_x)_x.$$

where $D(u) = \mathrm{d}\mathrm{d}_v f(u, h(u))\,d(u)$. Since $\mathrm{d}^2_{vv}\mathcal{E}$ is symmetric, the relation (35) can be written as

$$\langle \mathrm{d}_v f, \mathrm{d}^2\eta\,\mathrm{d}_v f\rangle = \mathrm{d}^2_{vv}\mathcal{E}\,\Omega$$

showing that stability condition $\Omega \geq 0$ (suggested by the Chapman–Enskog expansion) plus the convexity in the $v$ direction of the entropy $\mathcal{E}$ guarantee the convexity of the reduced entropy $\eta$.

## 4. Dissipative entropies in the $2 \times 2$ case

Next we turn to consider the $2 \times 2$ case under the assumption [H]

$$\left\{ \begin{array}{l} u_t + f(u,v)_x = 0 \\ v_t + g(u,v)_x = q(u,v) \end{array} \right. \qquad u, v \in \mathbb{R}. \tag{36}$$

We assume this system to be strictly hyperbolic, the eigenvalues of $\mathrm{d}F$ being

$$\lambda^{\pm} := \frac{\mathrm{d}_u f + \mathrm{d}_v g \pm \sqrt{(\mathrm{d}_u f - \mathrm{d}_v g)^2 + 4\,\mathrm{d}_u g\,\mathrm{d}_v f}}{2}.$$

The relaxed system is a scalar conservation law

$$u_t + f_*(u)_x = 0 \qquad f_*(u) := f(u, h(u)) \tag{37}$$

In this case a simple stability condition can be introduced:

$$\textit{subcharacteristic condition:} \qquad \det(\mathrm{d}F(u,h(u)) - \mathrm{d}f_*(u)\,I) \leq 0. \tag{38}$$

Condition (38) means that the relaxed equation (37) cannot propagate signals faster than the original system (36), i.e.:

$$\lambda^-(u,h(u)) \leq \mathrm{d}f_*(u) \leq \lambda^+(u,h(u)).$$

Moreover, the dispersion relation of the linearized system at $(u, h(u))$ is

$$\lambda^2 + (\mathrm{Tr}(\mathrm{d}F)\mu - \mathrm{d}_v q)\lambda + (\det(\mathrm{d}F)\,\mu - \mathrm{d}_v q\,\mathrm{d}f_*)\;\mu = 0.$$

As $\mu \to 0$, the branch passing through $(0,0)$ has the expansion

$$\lambda = \lambda(\mu) = -\mathrm{d}f_*\,\mu + \frac{1}{\mathrm{d}_v q}\,\det(\mathrm{d}F - \mathrm{d}f_*\,I)\,\mu^2 + o(\mu^2).$$

Thus (recalling that $\mathrm{d}_v q < 0$) (38) is a necessary condition for linearized stability of the steady state $(u, h(u))$.

In the $2 \times 2$ case, there holds

$$\det(\mathrm{d}F - \mathrm{d}f_* I) = \mathrm{d}_v f(\mathrm{d}h\,\mathrm{d}f_* - \mathrm{d}g_*). \tag{39}$$

Moreover, condition [SK] is equivalent to the request

$$\mathrm{d}h\,\mathrm{d}f_* - \mathrm{d}g_* \neq 0.$$

Thus, for a $2 \times 2$ system satisfying the subcharacteristic condition, the strict inequality in (38) is satisfied if and only if the Shizuta–Kawashima condition [SK] holds and $\mathrm{d}_v f \neq 0$.

Since we are dealing with cases where condition [SK] is not satisfied, we are forced to consider the case in which the equality holds in (38). In this case, condition (35) implies

$$(\mathrm{d}_v f)^2\,\mathrm{d}^2\eta = -\mathrm{d}^2_{vv}\mathcal{E}\,\det(\mathrm{d}F - \mathrm{d}f_*\,I) = 0. \tag{40}$$

In particular, if $\mathrm{d}_v f \neq 0$ and [SK] does not hold, then $\mathrm{d}^2\eta = 0$ and, as a consequence, $L^2$ stability cannot hold. The other way round, *whenever (38) holds and* [SK] *does not, a necessary condition to have strictly convex entropies/$L^2$ stability is $\mathrm{d}_v f = 0$ on the equilibrium manifold.*

We want to show a converse result: namely, we want to show that condition (38), together with an additional assumption related to the coupling of the system, implies existence of dissipative entropies.

Given a $2 \times 2$ system (36) such that [H] holds, let

$$A := \{u : \det(\mathrm{d}F(u, h(u)) - \mathrm{d}f_*(u)\, I) = 0\}$$

and assume that the boundary $\partial A$ of $A$ is finite.

If the subcharacteristic condition (38) is satisfied and

$$\lim_{u \to \bar{u}, u \notin A} \frac{(\mathrm{d}_v f)^2}{\det(\mathrm{d}F - \mathrm{d}f_*\, I)} < 0 \qquad \forall\, \bar{u} \in \partial A, \tag{41}$$

then there exists an entropy $\mathcal{E}$ such that $\mathrm{d}_v\mathcal{E} = 0$ and $\mathrm{d}^2_{vv}\mathcal{E} > 0$ at the equilibrium (hence $\mathcal{E}$ is a dissipative entropy).

Thanks to (39), condition (41) can be rewritten as

$$\lim_{u \to \bar{u}, u \notin A} \frac{\mathrm{d}_v f}{\mathrm{d}h\, \mathrm{d}f_* - \mathrm{d}g_*} < 0 \qquad \forall\, \bar{u} \in \partial A,$$

showing that the functions $\mathrm{d}_v f$ (related to the coupling of the system) and $\mathrm{d}h\, \mathrm{d}f_* - \mathrm{d}g_*$ (related to condition [SK]) must have the same order of zero on the equilibrium manifold whenever equality in (38) holds.

**Proof of Theorem 4.** The entropy $\mathcal{E}$, in the $2 \times 2$ case, solves

$$\mathcal{L}\,\mathcal{E} := \mathrm{d}_v f\, \mathrm{d}^2_{uu}\mathcal{E} + (\mathrm{d}_v g - \mathrm{d}_u f)\ \mathrm{d}^2_{uv}\mathcal{E} - \mathrm{d}_u g\, \mathrm{d}^2_{vv}\mathcal{E} = 0.$$

together with the conditions at equilibirum

$$\mathcal{E}\big|_\gamma = \eta(u), \qquad \mathrm{d}_v\mathcal{E}\big|_\gamma = 0 \qquad \forall\, u.$$

where $\gamma : \mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $\gamma(u) := (u, h(u))$. Here $\eta$ is an arbitrary strictly convex function; the second condition being necessary in order to build a dissipative entropy. Setting

$$\tilde{\mathcal{E}}(u, v) := \mathcal{E}(u, v) - \eta(u),$$

then

$$\mathcal{L}\,\tilde{\mathcal{E}} = -\mathcal{L}\,\eta(u) = -\mathrm{d}_v f(u, v)\, \mathrm{d}^2\eta(u), \qquad \tilde{\mathcal{E}}\big|_\gamma = 0, \qquad \tilde{\mathrm{d}}_v\mathcal{E}\big|_\gamma = 0.$$

Next we change variables:

$$\tilde{\mathcal{E}}(u, v) = E(x, y) \qquad \text{where} \quad \begin{cases} x = x(u, v), \\ y = y(u, v), \end{cases}$$

where $x$ and $y$ are chosen so that $(\mathrm{d}_u x, \mathrm{d}_v x) \neq (0, 0)$ and $(\mathrm{d}_u y, \mathrm{d}_v y) \neq (0, 0)$ for any $(u, v)$ under consideration and

$$\begin{cases} \mathrm{d}_v f(\mathrm{d}_u x)^2 + (\mathrm{d}_v g - \mathrm{d}_u f)\, \mathrm{d}_u x\, \mathrm{d}_v x - \mathrm{d}_u g(\mathrm{d}_v x)^2 = 0, \\ \mathrm{d}_v f(\mathrm{d}_u y)^2 + (\mathrm{d}_v g - \mathrm{d}_u f)\, \mathrm{d}_u y\, \mathrm{d}_v y - \mathrm{d}_u g(\mathrm{d}_v y)^2 = 0. \end{cases} \tag{42}$$

From (42) it follows that $(\mathrm{d}_u x, \mathrm{d}_v x)$ and $(\mathrm{d}_u y, \mathrm{d}_v y)$ are proportional to $(f_u - \lambda^\pm, \mathrm{d}_v f)$. Indeed, setting $(\mathrm{d}_u x, \mathrm{d}_v x) \propto (\mathrm{d}_u f - \lambda, \mathrm{d}_v f)$ in (42) we get

$$\begin{aligned} 0 &= (\mathrm{d}_u f - \lambda)^2 + (\mathrm{d}_v g - \mathrm{d}_u f)\, (\mathrm{d}_u f - \lambda) - \mathrm{d}_v f\, \mathrm{d}_u g \\ &= \lambda^2 - (\mathrm{d}_u f + \mathrm{d}_v g)\lambda + \mathrm{d}_u f\, \mathrm{d}_v g - \mathrm{d}_v f\, \mathrm{d}_u g. \end{aligned}$$

Denoting by $\Gamma = \{(x(u, h(u)), y(u, h(u)))\}$, the function $E$ solves

$$\begin{cases} L\,E := \mathrm{d}^2_{xy}E + b_1\, \mathrm{d}_x E + b_2\, \mathrm{d}_y E = F := \dfrac{\mathrm{d}_v f\, \mathrm{d}^2\eta}{(\lambda^+ - \lambda^-)^2}, \\ \\ E\big|_\Gamma = 0, \qquad \mathrm{d}_v x\, \mathrm{d}_x E + \mathrm{d}_v y\, \mathrm{d}_y E\big|_\Gamma = 0 \end{cases} \tag{43}$$

where $b_1$ and $b_2$ are appropriately defined. Differentiating the first of the two conditions at $\Gamma$ and considering it together with the second initial condition in problem (43), we get a linear system for $\mathrm{d}E$

$$\begin{pmatrix} \mathrm{d}_u x + \mathrm{d}_v x\, \mathrm{d}h & \mathrm{d}_u y + \mathrm{d}_v y\, \mathrm{d}h \\ \mathrm{d}_v x & \mathrm{d}_v y \end{pmatrix} \mathrm{d}E = 0$$

Since the determinant of the coefficients matrix is $\lambda^+ - \lambda^- \neq 0$, at the equilibrium, there holds $\mathrm{d}E\big|_\Gamma = 0$. Thus the problem (43) can be rewritten as

$$\begin{cases} L\,E := \mathrm{d}^2_{xy} E_{xy} + b_1\,\mathrm{d}_x E + b_2\,\mathrm{d}_y E = F, \\ \\ E\big|_\Gamma = 0, \qquad \mathrm{d}_y E\big|_\Gamma = 0 \end{cases} \tag{44}$$

The curve $\Gamma$ in the $(x, y)$-plane is parametrized by $u$

$$\Gamma(u) = (f_*^-(u), f_*^+(u)) \tag{45}$$

where

$$f_*^-(u) := x(u, h(u)),\ f_*^+(u) := y(u, h(u))$$

From definitions of $f_*^\pm$, it follows

$$\mathrm{d}f_*^-(u) = \mathrm{d}_u f + \mathrm{d}_v f\,\mathrm{d}h - \lambda^- = \mathrm{d}f_* - \lambda^-,$$
$$\mathrm{d}f_*^+(u) = \mathrm{d}_u f + \mathrm{d}_v f\,\mathrm{d}h - \lambda^+ = \mathrm{d}f_* - \lambda^+.$$

If condition (38) is satisfied with strict inequlity, functions $f_*^\pm$ are strictly monotone, hence invertible. In this case, it is possible to solve (44) by constructing a so-called Riemann function (see [14]). Since we are assuming only the subcharacteristic condition (38), functions $f_*^\pm$ are (weakly) monotone and thus have pseudoinverses: $(f_*^\pm)^{-1}$. The construction of the Riemann function can be extended to this wider framework with special care to the points where either $(f_*^-)^{-1}(x)$ or $(f_*^+)^{-1}(y)$ is not a singleton.

Assume $x \geq f_*^-((f_*^+)^{-1}(y))$ and $y \geq f_*^+((f_*^-)^{-1}(x))$ The case $x \leq f_*^-((f_*^+)^{-1}(y))$ and $y \leq f_*^+((f_*^-)^{-1}(x))$ can be treated similarly (mixed situations are prohibited by the subcharacteristic condition (38)). Then, for any $P = (x, y)$ such that $(f_*^-)^{-1}(x)$, $(f_*^+)^{-1}(y)$ are a singletons, the following set is well-defined

$$\Omega_{(x,y)} := \{(\xi, \zeta)\ :\ \xi \in [f_*^-((f_*^+)^{-1}(y)), x],\ \zeta \in [f_*^+((f_*^-)^{-1}(\xi)), y]\}$$
$$\equiv \{(\xi, \zeta)\ :\ \zeta \in [f_*^+((f_*^-)^{-1}(x)), y],\ \xi \in [f_*^-((f_*^+)^{-1}(\eta)), x]\}$$

If either $(f_*^-)^{-1}(x)$ or $(f_*^+)^{-1}(y)$ is multivalued (hence they are nontrivial intervals) $\Omega_{(x,y)}$ is similarly defined with $[f_*^-((f_*^+)^{-1}(y)), x]$ defined by

$$[f_*^-((f_*^+)^{-1}(y)), x] := \bigcup_{f_*^+(a) = y} [f_*^-(a), x].$$

Denoting by $R = R(x, y; \xi, \eta)$ the Riemann function of the problem, the solution is given by

$$\begin{aligned} E(x, y) &= \iint_{\Omega_{(x,y)}} R(x, y; \xi, \zeta) F(\xi, \eta)\ d\xi\, d\zeta \\ &\equiv \int_{\Omega_{(x,y)}} S(x, y; \xi, \zeta)\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u(\xi, \zeta))\, d\xi\, d\zeta \end{aligned} \tag{46}$$

where

$$S(x,y;\xi,\zeta) := \frac{R(x,y;\xi,\zeta)}{(\lambda^+-\lambda^-)^2(\xi,\zeta)}$$

Going back to variables $(u,v)$, we get the following expression

$$\mathcal{E}(u,v) = \eta(u) + \int_{\Sigma_{(u,v)}} \frac{1}{\lambda^+-\lambda^-}\, R\big(x,y;\sigma,\tau\big)\mathrm{d}_v f(\sigma,\tau)\,\mathrm{d}^2\eta(\sigma)\,d\sigma\,d\tau \tag{47}$$

where $x = x(u,v)$ and $y = y(u,v)$. The function $E$ in (46) is well-defined for $P$ such that $(f_*^-)^{-1}(x)$, $(f_*^+)^{-1}(y)$ are singleton and for $P$ such that one of the two (or both) is an interval. Indeed, the presence of a multivalued function modifies the integral by a zero-measure set, hence the function is continuous at any such point.

If $(x,y)$ is such that $(f_*^-)^{-1}(x)$, $(f_*^+)^{-1}(y)$ are singletons:

$$\frac{\partial E}{\partial x} = \iint_\Omega \frac{\partial S}{\partial x}\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u)\,d\xi\ d\zeta + \int_{f_*^+}^{y} S(x,y;x,\zeta)\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u(x,\zeta))\,d\zeta.$$

$$\frac{\partial E}{\partial y} = \iint_\Omega \frac{\partial S}{\partial y}\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u)\,d\xi\ d\zeta + \int_{f_*^-}^{x} S(x,y;\xi,y)\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u(\xi,y))\,d\xi$$

where $f_*^+ = f_*^+((f_*^-)^{-1}(x))$ and $f_*^- = f_*^-((f_*^+)^{-1}(y))$. Continuity of the double integrals follows from considerations similar to the ones of the continuity analysis. To prove that $E \in C^1$ it is enough to show that also the line integrals are continuous at any point where either $(f_*^-)^{-1}(x)$ or $(f_*^+)^{-1}(y)$ is not a singleton. Let us consider the first one (the second can be dealt with similarly). Let $x_0$ be such that $f_*^-(u) = x_0$ for any $u \in [u_1,u_2]$ with $u_1 < u_2$. Let $x$ and $y$ be such that

$$(\mathrm{d}_u x, \mathrm{d}_v x) = \left(\frac{\mathrm{d}_u f - \lambda^-}{\mathrm{d}_v f}, 1\right), \qquad (\mathrm{d}_u y, \mathrm{d}_v y) = (\mathrm{d}_u f - \lambda^+, \mathrm{d}_v f).$$

for any $u \in (u_1-\varepsilon, u_1] \cup [u_2, u_2+\varepsilon)$ for some $\varepsilon > 0$. Hence, there holds

$$\left[\frac{\partial E}{\partial x}\right](x_0,y) = \int_{f_*^+(u_1)}^{y} S\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u)\,d\zeta - \int_{f_*^+(u_2)}^{y} S\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u)\,d\zeta$$

$$= \int_{f_*^+(u_1)}^{f_*^+(u_2)} S\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u)\,d\zeta,$$

where $[\,\cdot\,](x_0,y)$ denotes the jump with respect to the variable $x$ at $(x_0,y)$. Condition (41) implies $\mathrm{d}_v f\,\mathrm{d}^2\eta(u) = 0$ whenever $\det(\mathrm{d}F - \mathrm{d}f_*\,I) = 0$, that is at any point where $(f_*^-)^{-1}(x)$ or $(f_*^+)^{-1}(y)$ is not a singleton. Thus $\left[\frac{\partial E}{\partial x}\right](x_0,y) = 0$. Similar considerations hold for $\dfrac{\partial E}{\partial y}$. Thus $E \in C^1$.

Next we consider the second order derivatives:

$$\frac{\partial^2 E}{\partial x^2} = \iint_\Omega \frac{\partial^2 S}{\partial x^2}\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u)\,d\xi\ d\zeta + 2\int_{f_*^+}^{y} \frac{\partial S}{\partial x}(x,y;x,\zeta)\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u(x,\zeta))\,d\zeta$$

$$+ \int_{f_*^+}^{y} \left\{ \frac{\partial S}{\partial \xi}(x,y;x,\zeta)\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u(x,\zeta)) + S(x,y;x,\zeta)\,\frac{\partial}{\partial x}\mathrm{d}_v f\,\mathrm{d}^2\eta(u(x,\zeta))\right\} d\zeta$$

$$-S(x,y;x,f_*^+)\,\mathrm{d}_v f\,\mathrm{d}^2\eta(u(x,f_*^+))(f_*^+)',$$

where $f_*^+$ is calculated at $(f_*^-)^{-1}(x)$. The first four terms can be treated as before. Concerning the last one, by definition of $f_*^\pm$, it follows

$$(f_*^+)' = \frac{\mathrm{d}f_*^+}{\mathrm{d}f_*^-} = \frac{\mathrm{d}_u y + \mathrm{d}_v y\, h'}{\mathrm{d}_u x + \mathrm{d}_v x\, h'} = \mathrm{d}_v f\, \frac{\mathrm{d}f_* - \lambda^+}{\mathrm{d}f_* - \lambda^-},$$

(here everything is calculated at $(f_*^-)^{-1}(x)$). Hence, there holds

$$\mathrm{d}_v f\,(f_*^+)' = \frac{(\mathrm{d}_v f)^2}{\det(\mathrm{d}F - \mathrm{d}f_*\, I)}\,(\mathrm{d}f_* - \lambda^+)^2.$$

Assumption (41) implies the existence of the limit, and, as a consequence, the existence of a continuous $E_{xx}$. Similar calculations hold for $E_{xy}$ and $E_{yy}$. Hence the function $E$ is $C^2$ and it is the requested solution. □

## 5. Stability of equilibrium solutions

In Section 4, we have shown that it is possible to build up strictly convex dissipative entropies even in cases where the Shizuta–Kawashima condition does not hold. In principle, such entropies permit to deduce $L^2$ estimates for perturbation of equilibrium solutions under suitable additional stability conditions (see Section 3). In this final Section, we are interested in finding some classes of system where the stability analysis of equilibrium solution can be completed. The analysis in Section 3 shows that, in order to complete this program, it is necessary to compensate the loss of dissipation with the presence of some linear degeneracy. This is encoded in the structure assumption on the (artificial) relaxation system we consider the present Section. As usual, we consider an $N \times N$ relaxation system for the uknown $\tilde w \in \mathbb{R}^N$:

$$\tilde w_t + F(\tilde w)_x = G(\tilde w). \tag{48}$$

We assume $\tilde w = (u, v) \in \mathbb{R}^n \times \mathbb{R}^p$ and
[A1] the function $G : \mathbb{R}^N \to \mathbb{R}^N$ has the form

$$G(w) = -Bw := \begin{pmatrix} 0_{n\times n} & 0_{n\times p} \\ 0_{p\times n} & b_{p\times p} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

with $b + b^T > 0$;
[A2] the system (48) has a quadratic entropy $\mathcal{E}(w) := \frac{1}{2}\langle E_0 w, w\rangle$ for some symmetric $E_0 > 0$ of the form

$$E_0 = \begin{pmatrix} E_{11} & 0_{n\times p} \\ 0_{p\times n} & E_{22} \end{pmatrix}.$$

Let $F = (f, g)$, then equilbrium solutions $W = (U, 0)$ satisfy

$$u_t + \mathrm{d}_u f(u,0)\, u_x = \mathrm{d}_u g(u,0)\, u_x = 0.$$

We assume the following assumption

$$f(u,v) = A_0\, u + \tilde f(u,v) \qquad \mathrm{d}_u g(u,0) = 0 \qquad \forall\, u. \tag{49}$$

where $\tilde f$ is such that, for any $M > 0$, there exists $C$, depending on $M$, such that

$$|\tilde f(u,v)| + |\mathrm{d}_u \tilde f(u,v)| + |\mathrm{d}^2_{uu} \tilde f(u,v)| + |\mathrm{d}^3_{uuu} \tilde f(u,v)| \le C\,|v|^2, \tag{50}$$

for any $(u,v)$ with $|(u,v)| \le M$. Since $E_0 \mathrm{d}F$ is symmetric, $E_{11}\mathrm{d}_v f = \mathrm{d}_u g^t E_{22}$, hence

$$\mathrm{d}_v f(u,0) = E_{11}^{-1} \mathrm{d}_u g^t(u,0) E_{22} = 0 \qquad \forall\, u.$$

Assumption (49) and (50) implies that, for any $M > 0$, there exists $C$, depending on $M$, such that

$$|\mathrm{d}_v f(u,v)| + |\mathrm{d}^2_{uv} f(u,v)| + |\mathrm{d}^3_{uuv} f(u,v)| \le C\,|v| \qquad \forall\,|(u,v)| \le M. \tag{51}$$

$$|\mathrm{d}_u g(u,v)| + |\mathrm{d}^2_{uu} g(u,v)| + |\mathrm{d}^3_{uuu} g(u,v)| \le C\,|v| \qquad \forall\,|(u,v)| \le M. \tag{52}$$

Hence the equilibrium solution are determined by the solution of the reduced *linear* system $u_t + A_0\, u_x = 0$.

The modulated entropy is

$$J(w;W) = \mathcal{E}(W+w) - \mathcal{E}(W) - \mathrm{d}\mathcal{E}(W)w = \frac{1}{2}\,\langle E_0 w, w\rangle = \mathcal{E}(w).$$

Let $\tilde{w} = W + w$ where $W$ is a fixed reference stationary solution, hence $F(W)_x = 0$. The perturbation equation for $w$ is

$$w_t + F(\tilde{w})_x + Bw = 0,$$

or, equivalently

$$w_t + \mathrm{d}F(\tilde{w})\, w_x + \mathrm{d}F(\tilde{w})W_x + Bw = 0. \tag{53}$$

Let $W = (U,0)$ be an equilibrium solution of the relaxation system (48) satisfiying assumptions `[A1]`–`[A2]`, (49), (50), (51), (52).

If $|U_x|_{W^{2,\infty}}$ and the initial perturbation $w_0$ are sufficiently small in $H^2$, then the following estimate holds for any $t$

$$|w|^2_{H^2} + \int_0^t |v|^2_{L^2}(\tau)\, d\tau \le C\,|w_0|^2_{H^2}$$

for some $C$ independent on $w_0$. In particular, the equilibrium solution $W = (U,0)$ is stable in $H^2$.

A simple example fitting in this class is

$$\begin{cases} u_t + \left(\dfrac{1}{2}\, v^2\right)_x = 0, \\ v_t + (u\, v)_x = -v. \end{cases} \tag{54}$$

The characteristic speeds of the system are $\lambda_\pm(u,v) := \frac{1}{2}\, u \pm \sqrt{v^2 + u^2/4}$ and the reduced flux $f_*$ is identically zero. Hence, for any $v \ne 0$ the strict subcharacteristic condition is satisfied, while at $v = 0$, i.e. along the equilibrium manifold, only the weak subcharacteristic condition holds. Indeed, any couple $(U,0)$, where $U$ is an arbitrary function of $x$ only, is an equilibrium solution. Note also, that $\mathrm{d}F$ of the system (54) is symmetric and a dissipative entropy $\mathcal{E}$ is given by $\mathcal{E}(u,v) = \frac{1}{2}(u^2+v^2)$.

If $r_\pm = r_\pm(u,v)$ denote right eigenvectors of $\mathrm{d}F$ of (54) relative to the eigenvalues $\lambda_\pm$, straightforward calculation leads to

$$\nabla\lambda_- \cdot r_- = 0 \qquad \Longleftrightarrow \qquad \lambda_-(u,v) = 0$$

$$\nabla\lambda_+ \cdot r_+ = 0 \qquad \Longleftrightarrow \qquad v\,\lambda_+(u,v) = 0$$

at any $(u,v)$ with $u > 0$. As a consequence, we see that the two characteristic fields of (54) are genuinely nonlinear in small neighborhood of $(u,v)$ with $v \ne 0$ and linearly degenerate at $v = 0$. It is the presence of such a linear behavior at the equilibrium manifold that compensate the loss of the Shizuta–Kawashima condition and permits to prove existence of global smooth solution. The general result stated in Theorem 5 is based on the same kind of relation between loss of dissipation and linear behavior of the reduced system.

**Proof of Theorem 5**. The proof is based on energy estimates.

Zero-th order estimate. The zero-th order estimates is encoded in the following equality obtained by taking the scalar product $\langle E_0 w, \cdot\rangle$ against the perturbation equation

Let $\mathcal{F}$ be such that $\mathrm{d}\mathcal{F} = \mathrm{d}\mathcal{E}\mathrm{d}F$. Then, since $\mathcal{F}(W)_x = \mathrm{d}\mathcal{E}(W)F(W)_x = 0$, there holds

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\langle E_0 w, w\rangle\Big) + \frac{\partial}{\partial x}\Big(\mathcal{F}(\tilde w) - \mathcal{F}(W) - \langle E_0 W, F(\tilde w) - F(W)\rangle\Big) + \langle w, E_0 B w\rangle$$
$$= -\langle E_0 w, \mathrm{d}F(\tilde w)\,\tilde w_x\rangle + \langle E_0 \tilde w, dF(\tilde w)\tilde w_x\rangle - \langle E_0 W_x, F(\tilde w) - F(W)\rangle$$
$$-\langle E_0 W, \mathrm{d}F(\tilde w)\tilde w_x\rangle = -\langle E_0 W_x, F(\tilde w) - F(W)\rangle.$$

Since $E_0$ and $E_0\mathrm{d}F$ are symmetric,

$$\langle E_0 W_x, \mathrm{d}F(W)w\rangle = \langle W_x, E_0\mathrm{d}F(W)w\rangle = \langle E_0\mathrm{d}F(W)\,W_x, w\rangle = 0,$$

the previous relation can be written as

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\langle E_0 w, w\rangle\Big) + \frac{\partial K}{\partial x} + \langle w, E_0 B w\rangle \tag{55}$$
$$= -\langle W_x, E_0(F(\tilde w) - F(W) - \mathrm{d}F(W)w)\rangle.$$

where

$$K := \mathcal{F}(\tilde w) - \mathcal{F}(W) - \langle E_0 W, F(\tilde w) - F(W)\rangle.$$

There holds

$$\langle W_x, E_0(F(\tilde w) - F(W) - \mathrm{d}F(W)w)\rangle$$
$$= \langle U_x, E_{11}(f(\tilde w) - f(W) - \mathrm{d}_u f(W)u - \mathrm{d}_v f(W)v)\rangle$$
$$= \langle U_x, E_{11}(\tilde f(\tilde w) - \mathrm{d}_v \tilde f(W)v)\rangle,$$

thus

$$|\langle W_x, E_0(F(\tilde w) - F(W) - \mathrm{d}F(W)w)\rangle| \le C\,|U_x|_{L^\infty}\,|v|^2.$$

Hence, integrating (55) in $\mathbb{R}\times(0,t)$ and using assumption (50), we get, **for** $|U_x|_{L^\infty}$ **sufficiently small**,

$$|w|^2_{L^2} + C\int_0^t |v|^2_{L^2}(\tau)\,d\tau \le C|w_0|^2_{L^2}. \tag{56}$$

where the (positive) constants $C$ depends on the $L^\infty$-norm of $w$.

First order estimate. Setting $z = w_x$ and differentiating (53), we obtain

$$z_t + (F(\tilde w)z)_x + dF(\tilde w)_x W_x + dF(\tilde w)W_{xx} + Bz = 0. \tag{57}$$

Taking the scalar product against $E_0\,z$, we get

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\,\langle E_0\,z, z\rangle\Big) + \langle E_0 z, (F(\tilde w)z)_x\rangle + \langle E_0 z, dF(\tilde w)_x W_x\rangle$$
$$+\langle E_0 z, dF(\tilde w)W_{xx}\rangle + \langle z, E_0 B z\rangle = 0.$$

Thanks to (50), (51) and (52), there holds

$$|\langle E_0 z, dF(\tilde w)_x W_x\rangle| = |\langle u_x, E_{11}\mathrm{d}_u f(\tilde w)_x U_x\rangle + \langle v_x, E_{22}\mathrm{d}_u g(\tilde w)_x U_x\rangle|$$
$$\le C\,|U_x|_{L^\infty}(|v|^2 + |v_x|^2)$$
$$|\langle E_0 z, dF(\tilde w)W_{xx}\rangle| = |\langle U_{xx}, E_{11}(\mathrm{d}_u f(\tilde w)u_x + \mathrm{d}_v f(\tilde w)v_x\rangle| \le C\,|U_{xx}|_{L^\infty}(|v|^2 + |v_x|^2)$$

for some constant $C$ depending on the $W^{1,\infty}$-norm of $w$.

Hence, after integration and using (56), for $|U_x|_{W^{1,\infty}}$ sufficiently small,

$$|w|^2_{H^1} + C\int_0^t |v|^2_{H^1}(\tau)\,d\tau \leq C|w_0|^2_{H^1} + C\int_0^t\int_{\mathbb{R}} |\langle z, (E_0\,F(\tilde{w})z)_x\rangle|\,dx\,d\tau. \tag{58}$$

Since

$$\langle z, E_0\,F(\tilde{w})_x z\rangle = \langle u_x, E_{11}\,\mathrm{d}_u f(\tilde{w})_x u_x\rangle + \langle u_x, E_{11}\,\mathrm{d}_v f(\tilde{w})_x v_x\rangle + \langle v_x, E_{22}\,\mathrm{d}_u g(\tilde{w})_x u_x\rangle + \langle v_x, E_{22}\,\mathrm{d}_v g(\tilde{w})_x v_x\rangle,$$

using the identity

$$\langle z, (E_0\,F(\tilde{w})z)_x\rangle = \frac{\partial}{\partial x}\left(\frac{1}{2}\langle z, E_0\,F(\tilde{w})z\rangle\right) + \frac{1}{2}\langle z, E_0\,F(\tilde{w})_x z\rangle,$$

we get

$$\int_{\mathbb{R}} |\langle z, (E_0\,F(\tilde{w})z)_x\rangle|\,dx \leq C\int_{\mathbb{R}} |\langle z, E_0\,F(\tilde{w})_x z\rangle|\,dx \leq C(|U_x|_{L^\infty} + |w_x|_{H^1})|v|^2_{H^1}$$

Therefore, the following estimate holds

$$\int_0^t\int_{\mathbb{R}} |\langle z, (E_0\,F(\tilde{w})z)_x\rangle|\,dx\,d\tau \leq C\int_0^t (|U_x|_{L^\infty} + |w_x|_{H^1})|v|^2_{H^1}(\tau)\,d\tau.$$

Inserting in (58), we obtain, for $|U_x|_{W^{1,\infty}}$ sufficiently small,

$$|w|^2_{H^1} + C\int_0^t |v|^2_{H^1}(\tau)\,d\tau \leq C|w_0|^2_{H^1} + C\int_0^t |w_x|_{H^1}\,|v|^2_{H^1}(\tau)\,d\tau \tag{59}$$

<u>*Second order estimate.*</u> Setting $\phi = z_x = w_{xx}$ and differentiating (57),

$$\phi_t + (F(\tilde{w})z)_{xx} + \mathrm{d}F(\tilde{w})_{xx}W_x + 2\mathrm{d}F(\tilde{w})_x W_{xx} + \mathrm{d}F(\tilde{w})W_{xxx} + B\phi = 0. \tag{60}$$

Taking the scalar product against $E_0\phi$, we obtain

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\langle E_0\phi, \phi\rangle\Big) + \langle E_0\phi, (F(\tilde{w})z)_{xx}\rangle + \langle E_0\phi, \mathrm{d}F(\tilde{w})_{xx}W_x\rangle + 2\langle E_0\phi, \mathrm{d}F(\tilde{w})_x W_{xx}\rangle + \langle E_0\phi, \mathrm{d}F(\tilde{w})W_{xxx}\rangle + \langle \phi, E_0 B\phi\rangle = 0.$$

Similarly to analogous relations obtained in the first order estimates, we have

$$2|\langle E_0\phi, \mathrm{d}F(\tilde{w})_x W_{xx}\rangle| + |\langle E_0\phi, \mathrm{d}F(\tilde{w})W_{xxx}\rangle| \leq C\,|U_{xx}|_{W^{1,\infty}}(|v|^2 + |v_x|^2 + |v_{xx}|^2)$$

for some constant $C$ depending on the $W^{1,\infty}$-norm of $w$. Hence, for $|U_{xx}|_{W^{1,\infty}}$, small

$$|w|^2_{H^2} + C\int_0^t |v|^2_{H^2}(\tau)\,d\tau \leq C|w_0|^2_{H^2} + C\int_0^t\int_{\mathbb{R}} \{|\langle E_0\phi, (F(\tilde{w})z)_{xx}\rangle| + |\langle E_0\phi, \mathrm{d}F(\tilde{w})_{xx}W_x|\}\,d\tau \tag{61}$$

Moreover

$$\langle E_0\phi, \mathrm{d}F(\tilde{w})_{xx}W_x\rangle = \langle u_{xx}, E_{11}\mathrm{d}_u f(\tilde{w})_{xx}U_x\rangle + \langle v_{xx}, E_{22}\mathrm{d}_u g(\tilde{w})_{xx}U_x\rangle$$

Since

$$(\mathrm{d}_u f)_{xx} = \mathrm{d}^3_{uuu} f\, u_x^2 + 2\mathrm{d}^3_{uuv} f\, u_x v_x + \mathrm{d}^3_{uvv} f\, v_x^2 + \mathrm{d}^2_{uu} f\, u_{xx} + \mathrm{d}^2_{uv} f\, v_{xx}$$

with similar relation for $(\mathrm{d}_u g)_{xx}$, thanks to (50), (51) and (52), there holds

$$\int_{\mathbb{R}} |\langle E_0\phi, \mathrm{d}F(\tilde{w})_{xx} W_x\rangle| \le C\,(|U_{xx}|_{L^\infty} + |w_x|_{H^1})|v|^2_{H^2}. \tag{62}$$

Finally, since

$$\begin{aligned}\langle w_{xx}, (E_0 \mathrm{d}F(\tilde{w}) w_x)_{xx}\rangle = \frac{\partial}{\partial x}\left(\frac{1}{2}\,\langle w_{xx}, E_0 \mathrm{d}F(\tilde{w}) w_{xx}\rangle\right)\\ + \langle w_{xx}, E_0 \mathrm{d}F(\tilde{w})_{xx} w_x\rangle + \frac{3}{2}\langle w_{xx}, E_0 \mathrm{d}F(\tilde{w})_x w_{xx}\rangle\end{aligned}$$

there holds

$$\begin{aligned}\int_{\mathbb{R}} |\langle \phi, (E_0 \mathrm{d}F(\tilde{w}) z)_{xx}\rangle|\,dx \le C\int_{\mathbb{R}} |\langle \phi, E_0 \mathrm{d}F(\tilde{w})_{xx} z\rangle|\,dx\\ + C\int_{\mathbb{R}} |\langle \phi, E_0 \mathrm{d}F(\tilde{w})_x \phi\rangle|\,dx.\end{aligned}$$

Since

$$\begin{aligned}\langle \phi, E_0\,\mathrm{d}F(\tilde{w})_x \phi\rangle = \langle u_{xx}, E_{11}\,\mathrm{d}_u f(\tilde{w})_x u_{xx}\rangle + \langle u_{xx}, E_{11}\,\mathrm{d}_v f(\tilde{w})_x v_{xx}\rangle\\ + \langle v_{xx}, E_{22}\,\mathrm{d}_u g(\tilde{w})_x u_{xx}\rangle + \langle v_{xx}, E_{22}\,\mathrm{d}_v g(\tilde{w})_x v_{xx}\rangle,\end{aligned}$$

we get

$$\int_{\mathbb{R}} |\langle \phi, E_0\,\mathrm{d}F(\tilde{w})_x \phi\rangle|\,dx \le C(|U_x|_{L^\infty} + |w_x|_{H^1})|v|^2_{H^2}$$

Finally

$$\begin{aligned}\langle \phi, E_0 \mathrm{d}F(\tilde{w})_{xx} z\rangle = \langle u_{xx}, E_{11}\mathrm{d}_u f(\tilde{w})_{xx} u_x + E_{11}\mathrm{d}_v f(\tilde{w})_{xx} v_x\rangle\\ + \langle v_{xx}, E_{22}\mathrm{d}_u g(\tilde{w})_{xx} u_x + E_{22}\mathrm{d}_v g(\tilde{w})_{xx} v_x\rangle.\end{aligned}$$

hence (using once more estimates (50), (51) and (52)),

$$\int_{\mathbb{R}} |\langle \phi, E_0 \mathrm{d}F(\tilde{w})_{xx} z\rangle|\,dx \le C(|U_x|_{L^\infty} + |w_x|_{H^1})|v|^2_{H^2}.$$

Therefore, the following estimate holds

$$\int_0^t\int_{\mathbb{R}} |\langle \phi, (E_0 \mathrm{d}F(\tilde{w}) z)_{xx}\rangle|\,dx\,d\tau \le C\int_0^t (|U_x|_{L^\infty} + |w_x|_{H^1})|v|^2_{H^2}(\tau)\,d\tau. \tag{63}$$

Inserting (62) e (63) in (61), we get, for $|U_x|_{W^{2,\infty}}$ sufficiently small,

$$|w|^2_{H^2} + C\int_0^t |v|^2_{H^2}(\tau)\,d\tau \le C|w_0|^2_{H^2} + C\int_0^t |w_x|_{H^1}\,|v|^2_{H^2}(\tau)\,d\tau \tag{64}$$

*Closing the argument.* Let $\eta(t) := \sup\{E(s)\,:\,s \in [0,t]\}$ where

$$E(t) := |w|^2_{H^2} + \int_0^t |v|^2_{L^2}(\tau)\,d\tau.$$

Then for any $s \in [0,t]$

$$E(s) \le C|w_0|^2_{H^2} + C\sup_s |w_x|_{H^1}\int_0^t |w_x|_{H^1}|v|^2_{H^2}\,d\tau.$$

Therefore

$$\eta(t) \le C|w_0|^2_{H^2} + C\eta^{3/2}(t).$$

As a consequence, if $w_0$ is sufficiently small, $\eta(t) \le C\,|w_0|^2_{H^2}$ for some $C$ independent on $w_0$.

## 6. Appendix

Here, we present how the whole picture appears under the unrealistic assumption of total dissipation. The aim is mainly didactical: we want to show in these easy cases all of the main feature of the problem and the basic strategy that can be used to approach it.

Let us start with the simplest example: the multidimensional *scalar* conservation law with damping, i.e.

$$w_t + \mathrm{div}_x\, F(w) + \delta w = 0, \qquad \delta \ge 0. \tag{65}$$

wher $F \in C^1$ with $\mathrm{d}f$ Lipschitz continuous, and $\delta > 0$. In the case $\delta = 0$ and $F$ not affine, there exist small perturbations of constant states developing a shock in finite time. What does it happen if $\delta > 0$? Shock formation is due to the intersection of characteristics in finite time. Starting from the initial datum $w(x,0) = w_0(x)$, we solve the characteristic system obtaining

$$x(t;x_i) = x_i + \int_0^t \mathrm{d}f(w_0(x_i)e^{-\delta\tau})\,d\tau,$$

where $x(\cdot;x_i)$ be the characteristic curve from $(x_i,0)$. Then, for $x_1 \ne x_0$,

$$\frac{x(t;x_1) - x(t;x_0)}{|x_1 - x_0|} = \frac{x_1 - x_0}{|x_1 - x_0|} + \int_0^t \frac{\mathrm{d}f(w_0(x_1)e^{-\delta\tau}) - \mathrm{d}f(w_0(x_0)e^{-\delta\tau})}{|x_1 - x_0|}\,d\tau.$$

Since

$$\left|\int_0^t \frac{\mathrm{d}f(w_0(x_i)e^{-\delta\tau}) - \mathrm{d}f(w_0(x_i)e^{-\delta\tau})}{|x_1 - x_0|}\,d\tau\right| \le \frac{1}{\delta}\,\mathrm{Lip}\,(w_0)\,\mathrm{Lip}\,(\mathrm{d}f)$$

if the initial datum $w_0$ is chosen so that $\mathrm{Lip}\,(w_0) < \delta/\mathrm{Lip}\,(\mathrm{d}f)$, then the characteristics $x(\cdot;x_0)$ and $x(\cdot;x_1)$ do not intersect for any positive time. Hence, *if $w_0$ is such that*

$$\mathrm{Lip}\,(w_0) < \frac{\delta}{\mathrm{Lip}\,(\mathrm{d}f)}$$

*then the solution to the Cauchy problem (65), $w(x,0) = w_0(x)$ is globally smooth.*

Note that the stronger the dissipation is (that is the larger $\delta$ is)/the weaker the nonlinearity of $F$ is (that is the smaller $\mathrm{Lip}\,(\mathrm{d}f)$ is) the weaker is the assumption on the initial data. The case $F$ linear should be considered as a limit case: if $\mathrm{Lip}\,(\mathrm{d}f) = 0$, then any smooth initial datum gives raise to global smooth solution.

In the case of systems of balance laws, similar conclusion on shock formation can be obtained through energy estimates, thanks to the Sobolev embedding $H^{[\frac{d}{2}]+1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$. For simplicity, let us consider the one–dimensional hyperbolic system:

$$w_t + F(w)_x = G(w),$$

assuming $F \in C^3$, $G \in C^2$, and, for some $\delta > 0$,

$$G(0) = 0, \qquad \langle \mathrm{d}G(w)\,\xi, \xi\rangle \le -\delta|\xi|^2 \qquad \forall w, \xi \in \mathbb{R}^n. \tag{66}$$

Additionally, to simplify the presentation, let us assume that $F(w) = dH(w)$ for some $H \in C^2(\mathbb{R}^N, \mathbb{R})$, so that $\mathrm{d}F(w) = \mathrm{d}^2 H(w)$ is a symmetric $N \times N$–matrix. Setting $G = (G_1, \dots, G_N)$,

$$\langle w, G(w) - G(0)\rangle = \int_0^1 \langle w, \mathrm{d}G(tw)w\rangle\, dt \le -\delta |w|^2.$$

Hence multiplying the equation by $w$ and setting $\mathcal{F}(w) := \langle w, F(w)\rangle - H(w)$, we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|w|^2\right) + \frac{\partial}{\partial x}\mathcal{F}(w) + \delta |w|^2 \le 0.$$

Integrating over $\mathbb{R} \times [0, t]$, we get

$$|w|^2_{L^2}(t) + 2\delta \int_0^t |w|^2_{L^2}(\tau)\, d\tau \le |w_0|^2_{L^2} \tag{67}$$

Similarly we can derive first and second order estimates. Differentiating the equation for $w$, setting $z = w_x$, and taking the scalar product against $z$,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|z|^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\langle z, \mathrm{d}F(w)\, z\rangle\right) + \frac{1}{2}\langle z, \partial_x(\mathrm{d}F(w))z\rangle - \langle z, \mathrm{d}G(w)\, z\rangle = 0.$$

Integrating over $\mathbb{R} \times [0, t]$, we obtain

$$|w_x|^2_{L^2}(t) + 2\delta \int_0^t |w_x|^2_{L^2}(\tau)\, d\tau \le |w_0'|^2_{L^2} + C\int_0^t |w_x|_{L^\infty} |w_x|^2_{L^2}(\tau)\, d\tau. \tag{68}$$

where $C$ denotes a constant (depending on the $L^\infty$–norm of $w$) bounding $\mathrm{d}^2F$.

Differentiating once more the equation for $w$ and setting $\phi = z_x = w_{xx}$,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\phi|^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\langle \phi, \mathrm{d}F(w)\, \phi\rangle\right) + \frac{3}{2}\langle \psi, \partial_x(\mathrm{d}F(w))\psi\rangle$$
$$+\langle \psi, \partial_x[\partial_x(\mathrm{d}F(w)) - \mathrm{d}G(w)]z\rangle - \langle \psi, \mathrm{d}G(w)\, \psi\rangle = 0.$$

Again, integrating over $\mathbb{R} \times [0, t]$,

$$|w_{xx}|^2_{L^2}(t) + 2\delta \int_0^t |w_{xx}|^2_{L^2}(\tau)\, d\tau \le |w_0''|^2_{L^2} + b\int_0^t |w_x|_{L^\infty} |w_x|^2_{H^1}(\tau)\, d\tau \tag{69}$$

where the constant $C$ (depending on the $L^\infty$–norm of $w$) bounds $\mathrm{d}^2F, \mathrm{d}^3F$ and $\mathrm{d}^2G$.

Finally, putting together estimates (67)–(68)–(69), we get

$$|w|^2_{H^2}(t) + 2\delta \int_0^t |w|^2_{H^2}(\tau)\, d\tau \le |w_0|^2_{H^2} + C\int_0^t |w_x|_{L^\infty} |w_x|^2_{H^1}(\tau)\, d\tau$$

By Morrey's Theorem, there exists $M > 0$ such that $|w_x|_{L^\infty} \le M|w_x|_{H^1} \le M|w|_{H^2}$, hence

$$E(t) := |w|^2_{H^2}(t) + 2\delta \int_0^t |w|^2_{H^2}(\tau)\, d\tau \le |w_0|^2_{H^2} + C\int_0^t |w|^3_{H^2}(\tau)\, d\tau \tag{70}$$

for some $C > 0$. Setting $\eta(t) = \sup_{s \in [0,t]} E(s)$, there holds $E(s) \le |w_0|^2_{H^2} + C\eta^{3/2}(t)$; thus

$$\eta(t) \le |w_0|^2_{H^2} + C\eta^{3/2}(t).$$

Hence there exists $C > 0$ such that, for $|w_0|^2_{H^2}$ small,

$$\eta(t) \le C|w_0|^2_{H^2} \qquad \forall\, t > 0$$

that is

$$\sup_{t>0} |w|^2_{H^2} + 2\delta \int_0^{+\infty} |w|^2_{H^2}(\tau)\, d\tau \le C|w_0|^2_{H^2},$$

showing the global boundedness of derivatives of $w$. From (67), exponential decay of the $L^2$ norm of $w$ immediately follows. A similar property can be deduced for the $H^2$-norm using the derivatives estimates.

## References


[1] Aregba-Driollet D.; Natalini R., *Discrete kinetic schemes for multidimensional systems of conservation laws*, SIAM J. Numer. Anal. 37 (2000), no.6, 1973–2004.

[2] Bianchini, S.; *A Glimm type functional for a special Jin-Xin relaxation model*, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001), no. 1, 19–42.

[3] Bianchini, S.; *Hyperbolic limit of the Jin-Xin relaxation model*, Comm. Pure Appl. Math. 59 (2006), no. 5, 688–753.

[4] Bianchini S.; Hanouzet B.; Natalini R.; *Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Comm. Pure Appl. Math., to appear.

[5] Bouchut F., *A reduced stability condition for nonlinear relaxation to conservation laws*, J. Hyperbolic Differ. Equ. 1 (2004), no. 1, 149–170.

[6] Brenier, Y.; Corrias, L.; *A kinetic formulation for multi-branch entropy solutions of scalar conservation laws*, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), no. 2, 169–190.

[7] Brenier, Y.; Corrias, L.; Natalini, R.; *Relaxation limits for a class of balance laws with kinetic formulation*, Advances in nonlinear partial differential equations and related areas (Beijing, 1997), 2–14, World Sci. Publ., River Edge, NJ, 1998.

[8] Carbou G.; Hanouzet B.; *Comportement semi-linéaire d'un systeme hyperbolique quasi-linéaire: le modle de -Debye* (French) [Semilinear behavior for a quasilinear hyperbolic system: the Kerr-Debye model] C. R. Math. Acad. Sci. Paris 343 (2006), no. 4, 243–247.

[9] Carbou G.; Hanouzet B.; *Relaxation approximation of some nonlinear Maxwell initial-boundary value problem*, Commun. Math. Sci. 4, no. 2 (2006), 331344

[10] Chen G.Q., Levermore C.D., Liu T.-P., *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. 47 (1994), no. 6, 787–830.

[11] Chern, I-L., *Long-time effect of relaxation for hyperbolic conservation laws*, Comm. Math. Phys. 172 (1995), no. 1, 39–55.

[12] Dafermos C.M., *A system of hyperbolic conservation laws with frictional damping*, Z. Angew. Math. Phys. 46 (1995), no. Special Issue, S294–S307, Theoretical, experimental, and numerical contributions to the mechanics of fluids and solids.

[13] Dafermos, C.; Hsiao, L.; *Hyperbolic systems and balance laws with inhomogeneity and dissipation*, Indiana Univ. Math. J. 31 (1982), no. 4, 471–491.

[14] DiBenedetto, E.; "Partial differential equations", Birkhäuser Boston, Inc., Boston, MA, 1995.

[15] Hanouzet, B.; Huynh, Ph.; *Approximation par relaxation d'un systeme de Maxwell non lineaire*, C. R. Acad. Sci. Paris Ser. I Math. 330 (2000), 193–198.

[16] Hanouzet B.; Natalini, R.; *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Ration. Mech. Anal. 169 (2003), no. 2, 89–117.

[17] Hosono T.; Kawashima S.; *Decay property of regularity-loss type and application to some hyperbolic-elliptic systems*, Math. Models Methods Appl. Sci. 16 (2006), no. 11, 1839–1859.

[18] Hsiao L.; Liu T.-P.; *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, Comm. Math. Physics 143 (1992), 599–605.

[19] Ide K.; Haramoto K.; Kawashima S.; *Decay property of the dissipative Timoshenko system*, (in preparation).

[20] Jin S.; Xin Z.; *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. 48 (1995), 235–276.

[21] Kawashima S.; *Dissipative Structure of Regularity-loss Type and Applications*, to appear in Benzoni S. (ed.) et al., Hyperbolic problems: Theory, numerics, applications. Proceedings

of the eleventh international conference on hyperbolic problems, Lyon (France), July 17-21, 2006. Berlin: Springer.
[22] Kawashima, S.; Yong, W.-A.; *Dissipative structure and entropy for hyperbolic systems of balance laws*, Arch. Ration. Mech. Anal. 174 (2004), no. 3, 345–364.
[23] Li, T.; *Global solutions of nonconcave hyperbolic conservation laws with relaxation arising from traffic flow*, J. Differential Equations 190 (2003), 131–149.
[24] Liu, H.; *The $L^p$ stability of relaxation rarefaction profiles*, J. Differential Equations 171 (2001), no. 2, 397–411.
[25] Liu T.-P.; *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. 108 (1987), no. 1, 153–175.
[26] Liu, T.-P.; Zeng, Y.; *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws*, Mem. Amer. Math. Soc. 125 (1997), no. 599, viii+120 pp.
[27] Mascia, C.; *Stability and instability issues for relaxation shock profiles*, to appear in Benzoni S. (ed.) et al., Hyperbolic problems: Theory, numerics, applications. Proceedings of the eleventh international conference on hyperbolic problems, Lyon (France), July 17-21, 2006. Berlin: Springer.
[28] Mascia, C.; Zumbrun, K.: *Pointwise Green's function bounds and stability of relaxation shocks*, Indiana Univ. Math. J. 51 (2002), no. 4, 773–904.
[29] Mascia, C.; Zumbrun, K.: *Stability of large-amplitude shock profiles of general relaxation systems*, SIAM J. Math. Anal. 37 (2005), no. 3, 889–913
[30] Natalini R.; *Recent results on hyperbolic relaxation problems*, Analysis of systems of conservation laws (Aachen, 1997), Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 128–198.
[31] Nishida T.; *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Département de Mathématique, Université de Paris-Sud, Orsay, 1978, Publications Mathématiques d'Orsay, No. 78-02.
[32] Ruggeri, T.; Serre, D.; *Stability of constant equilibrium state for dissipative balance laws system with a convex entropy*, Q. Appl. Math., 62 (2004), 163-179.
[33] Serre, D.; *Relaxations semi-lineaire et cinetique des systemes de lois de conservation*, (French) [Semilinear and kinetic relaxations of systems of conservation laws] Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000), no. 2, 169–192.
[34] Shizuta, Y.; Kawashima, S.; *Systems of equations of hyperbolic–parabolic type with applications to the discrete Boltzmann equation,* Hokkaido Math. J. 14 (1984) 435–457.
[35] Sideris, T.C.; Thomases, B.; Wang, D.; *Long time behavior of solutions to the 3D compressible Euler equations with damping*, Comm. Partial Differential Equations 28 (2003), no. 3-4, 795–816.
[36] Tzavaras, A.E.; *Relative entropy in hyperbolic relaxation*, Commun. Math. Sci. 3 (2005), no. 2, 119–132.
[37] Whitham, G.B.; "Linear and nonlinear waves", Wiley, New York (1974).
[38] Yong, W.-An; *Entropy and global existence for hyperbolic balance laws*, Arch. Ration. Mech. Anal. 172 (2004), 247–266.
[39] Zeng, Y., *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation*, Arch. Ration. Mech. Anal. 150 (1999), no. 3, 225–279.

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