

Convergence of Numerical Schemes for Viscosity Solutions to Integro-differential Degenerate Parabolic Problems arising in Financial Theory

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Summary. We study the numerical approximation of viscosity solutions for integro-differential, possibly degenerate, parabolic problems. Similar models arise in option pricing, to generalize the celebrated Black–Scholes equation, when the processes which generate the underlying stock returns may contain both a continuous part and jumps. Convergence is proven for monotone schemes and numerical tests are presented and discussed.

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1. Introduction

In this paper we study the numerical approximation of a class of semilinear strongly degenerate parabolic integro-differential Cauchy problems of the following form:

$$(1.1) \quad \begin{cases} \partial_t u - \mathcal{L}_{\mathcal{I}}(x, t, \mathcal{I}, \mathcal{D}, \mathcal{D}^2)u + H(x, t, \mathcal{D}u, \mathcal{I}u) = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

where u_0 is a continuous initial data, $\mathcal{L}_{\mathcal{I}}$ is a linear degenerate elliptic operator and H is a nonlinear first order operator. Here $\mathcal{I}u$ is an integral term given by

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$$(1.2) \quad \mathcal{I}u = \int_{\mathbb{R}^D} M(u(x+z, t), u(x, t)) \mu_{x,t}(dz),$$

where $\mu_{x,t}$ is a positive bounded measure and M is a function which is non decreasing in the first argument, $M(u, u) = 0$ and such that

$$M(u, v) - M(w, z) \leq c((u - w)_+ + |v - z|).$$

Problems in this form arise when considering a financial derivative constructed on an underlying asset given, as in Merton [31], by a jump-diffusion process instead of the classical diffusion dynamics, in order to have a more realistic description of the market than the one obtained with the Black–Scholes model [11]. In particular, using the original Black–Scholes formula, the implied volatilities vary for different strikes and maturities, producing the so-called volatility skew or smile. On the contrary, a suitable choice of the jump process parameters can allow the model to fit the observed volatility. For some recent works concerning jump-diffusion models see [1, 7, 10, 20, 27]. Another recent domain of interest is the analytical modeling of jumps in fixed-income markets, [8, 13, 14, 17, 18, 20, 33]. The choice of a jump diffusion model in this setting seems to be the natural one [25, 18]; it is worth noticing that this choice preserves the mean reverting behaviour of the interest rate curve. More direct numerical issues are considered in [6] and in the book by D. Tavella and C. Randall [35].

From a mathematical perspective, the problem of existence and uniqueness of solutions to these equations has been intensively studied in the framework of classical solutions and for uniformly parabolic operators, see [21–23] and references therein. Unfortunately, due to the possible degeneracy of the parabolic operator, typical of incomplete markets [30, 3, 4], and the presence of nonlinear terms, as for instance in Example 2.2, we have to consider weaker (not continuously differentiable) solutions. This last is not a mathematical complication, as the typical example of nonlinearity is given by the large investor model, in which the movements of big amounts of money by a single investor influence the market itself, in particular the interest rate. Moreover it is worth noticing that in the real world banks apply different rates for borrowing or lending and this also brings nonlinear effects in the model. A natural approach is given by viscosity solutions [15]. In this framework, the integro-differential problems have been first studied by Alvarez and Tourin [2]. In particular they studied the equation

$$\partial_t u + F(x, t, u, \mathcal{D}u, \mathcal{D}^2 u) = \mathcal{I}u,$$

where F is a pure differential second order elliptic operator, $F \in \mathcal{C}(\mathbb{R}^D \times (0, T] \times \mathbb{R} \times \mathbb{R}^D \times \mathcal{S}_D)$, and $\mathcal{I}u$ is defined as above. We follow their approach, using an extended notion of viscosity solutions and viscosity inequalities, considering that the classical theory of [15] was simply concerned with the

purely differential case. More general results and problems can be found in [3–5] and references therein.

A great deal has been done for the numerical approximation of viscosity solutions, starting from [16]. For second order problems let us refer to the fundamental paper by Barles and Souganidis [9], who first showed convergence results for a large class of numerical schemes to the solution of fully nonlinear second order elliptic or parabolic PDE. Following their tracks, we here extend their arguments to the class of numerical schemes for integro-differential problems.

Let us also recall that in the framework of linear problems with constant coefficients, this new integral term was already considered in [6]. In that paper the authors proposed to use an operator splitting method compared with the drawbacks of a pure Crank-Nicholson one. In that context, the method was shown to be quite effective: it had a lighter computational burden and allowed to couple the differential part, with an implicit finite difference method, and the integral part, with an FFT method. The FFT method requires a constant grid step, however it could diminish the numerical precision of the scheme in some areas; it is possible to overcome this difficulty using an asymptotic profile of the solution or a particular feature of the integral operator. A closer discussion of this method is done in Section 7. We here wish to underline that its rigorous assessing, as well as its extension to fully nonlinear strongly degenerate problems, which are the main objectives of our investigation, seems to be quite difficult and still has to be done.

Another difficult problem stems from the nonlocal nature of the integral term. It is necessary to truncate the problem domain on one hand, and the integral domain on the other. As $\mu_{x,t}$ is a bounded measure, for a fixed $\nu > 0$ we can choose a bounded computational domain D_ν for the integral term, such that

$$\left| \int_{\mathbb{R}^D} \mu_{x,t}(dz) - \int_{D_\nu} \mu_{x,t}(dz) \right| < \nu,$$

and we can consider a new problem with $\mathcal{I}_\nu u = \int_{D_\nu} M(u(x+z), u(x,t)) \mu_{x,t}(dz)$ instead of $\mathcal{I}u$; after that, we have to truncate the domain of the problem.

Unfortunately, due to the non-local nature of the integral term, once we have found a given domain, we still need to use some approximation of the solution in a larger computational domain. The common approach consists in replacing the original problem with an homogeneous one, i.e. without the integral term, or to use some asymptotic representation formula for the solution.

Here we try a different approach. First we show that our original problem can be well approximated by a pure differential problem with an artificial diffusion. We apply this remark to implement an effective numerical boundary

condition, giving as a consequence a full convergence result for the global approximation scheme.

The article is organized as follows: in Section 2 we briefly describe the basic financial models, and in particular we introduce some details of the nonlinear model for a large investor; in Section 3 we recall some basic definitions and results concerning viscosity solutions, while the main convergence result is given in Section 4; Section 5 is devoted to the study of the numerical computation of the integral term and the study of the scheme in the one dimensional case; in Section 6 we show that we can approximate the integro–differential operator by a suitable diffusive differential model. In Section 7 we focus on numerical simulations: after a short review of some recent numerical methods for the linear PIDE, we give a quite complete description of our explicit scheme. In particular Subsection 7.2 deals with the numerical boundary conditions and proves that, under proper assumptions on the measure $\mu_{x,t}$ and on the grid steps, the whole scheme is second order accurate.

We conclude, in Section 8, by presenting some numerical examples. First we deal with the classical Morton equation. A two dimensional problem is studied in Subsection 8.2. In Subsection 8.3, we consider the large investor nonlinear case.

2. The financial model - option pricing with jump-diffusion processes

For the basic notions on financial markets we refer to [12]. Following the seminal paper by Merton [31], we consider models which take into account abrupt price movements caused by exogenous events or information. The aim is to avoid the discrepancies between the results given by the standard Black–Scholes model [11] and empirical evidence. For instance it is well-known that the implied volatility, fitted by using historical data in the Black–Scholes formula, is not a constant, but depends on the strike price and on the expiration time. Actually, in addition to small variations from the trend, which may be described by a Brownian motion, the analysis of price evolution reveals sudden and rare breaks. Therefore, from a probabilistic point of view, it is natural to model such behavior by some point processes that count the occurrences of rare and random events. We then consider a market whose money market account evolves according to the differential equation

$$dS_t^0 = S_t^0 r dt,$$

where r stands for a deterministic interest rate, while $S = (S^1, \dots, S^D)$, the risky assets, are described by a stochastic differential system of jump-diffusion type:

$$dS_t^i = S_t^i \left[\mu_i dt + \sum_{j=1}^P \sigma_{ij} dW_t^j + \sum_{k=1}^M \gamma_{ik} dN_t^k \right], \quad i = 1, \dots, D.$$

Here $W_t = (W_t^1, \dots, W_t^P)$ is a P -dimensional Brownian motion, $N_t = (N_t^1, \dots, N_t^M)$ is a M -dimensional Poisson process.

Remark 2.1. If all the γ_{ik} are zero, and the functions μ and σ are constant, we have the standard Black–Scholes model, where the stock price follows a log-normal random walk:

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

It is well known that Black and Scholes assumed ideal conditions in the market: there are no arbitrage opportunities and the market is complete. Constructing directly the hedging strategy, that is given deterministically as a function of (S_t, t) , we get the arbitrage price of the European contingent claim (T, G) as the solution of the linear final value problem on $(0, +\infty) \times (0, T)$:

$$(2.1) \quad \begin{cases} -\partial_t V = \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 V + r \partial_S S V - r V, \\ V(S, T) = G(S). \end{cases}$$

Example 2.1. [Merton model] The prototype of jump-diffusion market has been first proposed by Merton [31]. The underlying process' dynamics are given by the following equation:

$$\frac{dS_t}{S_t} = (\mu - \lambda k) dt + \sigma dW_t + (\eta - 1) dN_t.$$

where dW_t is a standard Brownian motion, dN_t is a Poisson counting process of intensity λ , that is:

$$dN_t = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt. \end{cases}$$

Moreover we are assuming that η is a log-normally distributed jump amplitude with probability density:

$$(2.2) \quad \tilde{\Gamma}_\delta(\eta) = \frac{\exp(-\frac{1}{2}(\frac{\log \eta}{\delta})^2)}{\sqrt{2\pi} \delta \eta},$$

k is the expectation $\mathbb{E}(\eta - 1)$, and the Brownian and the Poisson processes are uncorrelated. Setting $\mathcal{J}V(S) = \lambda \left(\int_D V(S\eta) \tilde{\Gamma}_\delta(\eta) d\eta - V \right)$, we promptly obtain by the Ito formula the following pricing equation

$$(2.3) \quad \begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda k) S \frac{\partial V}{\partial S} - rV + \mathcal{J}V \\ = \frac{\partial V}{\partial t} + \mathcal{L}_{\mathcal{J}} V = 0, \end{aligned}$$

with $D = [0, +\infty)$ and the final condition

$$V(T, S_T) = V_T(S_T).$$

Following [3, 4, 36], it is possible to consider more sophisticated models, namely the *large investor* economy: let $\xi(V, \mathcal{D}V, \mathcal{J}V) = V - S\sigma\theta_0 \cdot \mathcal{D}V - \phi_0 \cdot \mathcal{J}V$ denote the amount of money invested in stocks by the agent, obtained choosing a proper replicating portfolio; then the interest rate r is influenced by the agents by means of ξ . Usually r is a non-increasing function of ξ , but we shall suppose that $r(\cdot, \xi)\xi$ is non decreasing with respect to ξ . In addition, we do not impose that r is continuous with respect to ξ at zero, in order to include interesting examples. For details about the parameters θ_0 and ϕ_0 we refer to [3].

In this setting the function V must solve the following quasi-linear final value problem:

$$(2.4) \quad \begin{cases} \partial_t V + L_{\mathcal{J}}V = H(S, t, V, \mathcal{J}V, \mathcal{D}V), \\ V(S, T) = G(S), \end{cases}$$

where we have simply rewritten equation (2.3) with the addition of the non-linear first order operator:

$$(2.5) \quad H(S, t, V, \mathcal{J}V, \mathcal{D}V) = r(S, t, \xi) \cdot \xi.$$

Obviously, if all the parameters of the model r , μ_i , σ_{ij} and γ_{ij} are deterministic function of (S, t) , the problem is linear and we obtain the so called *small investor economy*.

Example 2.2. [Completion of the market in the large investor model] It is easily proven that a jump-diffusion market is incomplete because of the arbitrage opportunities arising at the jump time (see [12]). To overcome the difficulty of pricing a derivative it is possible to complete the jump-diffusion market by adding another derivative on the same underlying asset. A standard approach is to add a call whose parameters are taken directly from the market, and not by Ito rule. Therefore we can suppose that our market is described by

$$\begin{cases} dS_t^0 = S_t^0 r(\mathbf{X}, t, \xi) dt, \\ d\mathbf{X} = \mathbf{X}\alpha dt + \mathbf{X}\beta dW_t + \mathbf{X}(\gamma - \mathbf{1})dN_t, \end{cases}$$

where $\mathbf{X}_t = \text{diag}(S_t, C_t)$ and the vectors of expectation α , volatility β , and jump γ are:

$$\alpha = \begin{pmatrix} \mu - \lambda k \\ \mu_C \end{pmatrix}, \beta = \begin{pmatrix} \sigma \\ \sigma_C \end{pmatrix}, \gamma = \begin{pmatrix} \eta \\ \eta_C \end{pmatrix}.$$

The jump amplitude γ is now lognormally distributed with density

$$\tilde{\Gamma}(\gamma) = \tilde{\Gamma}(\eta) \cdot \tilde{\Gamma}_C(\eta_C).$$

In this frameset the pricing equation is the extension of (2.3) to the multidimensional case:

$$\begin{cases} \partial_t V + L_{\mathcal{J}} V = H(S, t, V, \mathcal{J}V, \mathcal{D}V), \\ V(\mathbf{X}, T) = G(\mathbf{X}). \end{cases}$$

Here, the operator

$$L_{\mathcal{J}} V = \frac{1}{2} \text{tr}[(\mathbf{X}\beta)(\mathbf{X}\beta)^T \mathcal{D}^2 V] + \mathbf{X}[\alpha + \beta\theta] \cdot \mathcal{D}V - \phi \cdot \mathcal{J}V,$$

is linearly degenerate elliptic. Moreover H has the same form as (2.5), with X playing the role of S , and $\mathcal{J}V$ as previously with $D = [0, +\infty) \times [0, +\infty)$. We can note that in this case the diffusion matrix is degenerate, since

$$rk((\mathbf{X}\beta)(\mathbf{X}\beta)^T) < 2.$$

If we apply a change of variable in order to have diffusion only in one direction:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \cdot \begin{pmatrix} \log S \\ \log C \end{pmatrix},$$

with $\vartheta = \arctg \frac{\sigma_C}{\sigma}$, and proper coefficients A, B, C, D we obtain:

$$\begin{aligned} \partial_t u + \frac{1}{2}(\sigma^2 + \sigma_C^2) \partial_{xx}^2 u + A \partial_x u + B \partial_y u - \phi \mathcal{I}u \\ = r \left(x, y, t, u + C \partial_x u + D \partial_y u - \phi_0 \mathcal{I}u \right) \\ \times \left(u + C \partial_x u + D \partial_y u - \phi_0 \mathcal{I}u \right), \end{aligned} \quad (2.6)$$

where:

$$\begin{aligned} \mathcal{I}u = \lambda \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x + \xi, y + \zeta, t) \frac{\exp\left(-\frac{\xi^2 + \zeta^2}{2\delta^2}\right)}{2\pi\delta\rho} \right. \\ \left. \cdot \exp\left(-\frac{(\delta^2 - \rho^2)(\xi \sin \theta + \zeta \cos \theta)^2}{2\delta^2\rho^2}\right) d\xi d\zeta - u(x, y, t) \right). \end{aligned}$$

This example shows the need for a theory of strongly degenerate nonlinear parabolic operators in financial applications.

3. Backgrounds: viscosity solutions for parabolic nonlinear integro-differential operators

The pricing equation for our model could be written as:

$$(3.1) \quad \begin{cases} \partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2u) = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

As for the general theory of viscosity solutions for pure differential equations, we shall make some assumptions on the function $F \in \mathcal{C}(\mathbb{R}^D \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^D \times \mathcal{S}_D, \mathbb{R})$, where \mathcal{S}_D is the set of the symmetric $D \times D$ matrices.

F1 F is degenerate elliptic:

$$F(x, t, u, \mathcal{I}, p, \mathcal{X}) \geq F(x, t, u, \mathcal{I}, p, \mathcal{Y}), \text{ for } \mathcal{X} \leq \mathcal{Y};$$

F2 F is quasi-monotone with respect to u , uniformly with respect to the other variables:

$$F(x, t, u, \mathcal{I}, p, \mathcal{X}) \geq F(x, t, v, \mathcal{I}, p, \mathcal{X}) - \gamma(u - v), \quad u \geq v,$$

where $\gamma \in \mathcal{C}([0, \infty))$, $\gamma(0) = 0$, $\gamma(u) > 0$ as $u > 0$ and:

$$\int_0^\varepsilon \frac{du}{\gamma(u)} = \infty \quad \text{for all } \varepsilon > 0.$$

To rely the dependency of F on the integral term, we shall also assume, following [4],

F3 F is non-increasing with respect to \mathcal{I} :

$$F(x, t, u, \mathcal{I}, p, \mathcal{X}) \geq F(x, t, u, \mathcal{J}, p, \mathcal{X}), \text{ for } \mathcal{I} \leq \mathcal{J}.$$

Following [2–4], we can now give a modified notion of viscosity solutions, which makes use of the notion of upper and lower semijets \mathcal{P}^\pm , see [15].

Definition 3.1. Given a function u and a point $(x, t) \in \mathbb{R}^D \times [0, T]$, we say that:

$$\partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2u) \leq 0 \text{ (resp. } \geq 0)$$

in viscosity sense at (x, t) if one of the following equivalent condition is met:

(1) for all $(\tau, p, \mathcal{X}) \in \mathcal{P}^+u(x, t)$ (respectively $\mathcal{P}^-(x, t)$):

$$\tau + F(x, t, u(x, t), \mathcal{I}u(x, t), p, \mathcal{X}) \leq 0 \text{ (resp. } \geq 0);$$

(2) for each function $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^D \times [0, T])$ such that $u - \phi$ has a local maximum at (x, t) (respectively, a minimum):

$$\partial_t \phi(x, t) + F(x, t, u(x, t), \mathcal{I}\phi(x, t), \mathcal{D}\phi(x, t), \mathcal{D}^2\phi(x, t)) \leq 0 \text{ (resp. } \geq 0);$$

(3) for all test function $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^D \times [0, T])$ such that $u - \phi$ has a global strict maximum at (x, t) (respectively, a minimum) and $\phi(x, t) = u(x, t)$:

$$\partial_t \phi + F(x, t, \phi, \mathcal{I}\phi, \mathcal{D}\phi, \mathcal{D}^2\phi) \leq 0 \text{ (resp. } \geq 0)$$

holds in classical sense.

Here we denote by $\mathcal{C}^{2,1}(\mathbb{R}^D \times [0, T])$ the set of functions that are twice continuously differentiable with respect to $x \in \mathbb{R}^D$ and once with respect to $t \in [0, T]$. Let us notice that, the important difference between the purely differential case, treated in [15], and the integro-differential one is that the local continuity property of the semijets of semicontinuous functions does not imply the semicontinuity of the equation, because the new nonlocal term $\mathcal{I}u$ does not preserve semicontinuity in general. To overcome this difficulty, we have to define a new class of admissible functions.

Definition 3.2. A function $f(y, s; z)$ has an **upper** (resp. **lower**) μ -bound at (x, t) if there exist a neighborhood $V_{x,t}$ of (x, t) and a function $\Phi \in \mathcal{C}(\mathbb{R}^D) \cap L^1(\mathbb{R}^D; \mu_{x,t})$ such that:

- $\int \Phi(z) \mu_{y,s}(dz) \rightarrow \int \Phi(z) \mu_{x,t}(dz)$ if $(y, s) \rightarrow (x, t)$;
- $f(y, s; z) \leq \Phi(z)$ (resp. \geq) $\mu_{y,s}$ - a.e. z , for all $(y, s) \in V_{x,t}$.

Remark 3.1. Let f be a locally bounded function on $\mathbb{R}^D \times [0, T] \times \mathbb{R}^D$ which has an upper (resp., lower) μ -bound at (x, t) ; then:

$$\begin{aligned} \limsup_{(y,s) \rightarrow (x,t)} \int_{\mathbb{R}^D} f(y, s; z) \mu_{y,s}(dz) &\leq \int_{\mathbb{R}^D} \limsup_{(y,s) \rightarrow (x,t)} f(y, s; z) \mu_{x,t}(dz) \\ &\left(\text{resp. } \liminf_{(y,s) \rightarrow (x,t)} \int_{\mathbb{R}^D} f(y, s; z) \mu_{y,s}(dz) \right. \\ &\left. \geq \int_{\mathbb{R}^D} \liminf_{(y,s) \rightarrow (x,t)} f(y, s; z) \mu_{x,t}(dz) \right). \end{aligned}$$

In this way we can introduce some new classes of admissible functions.

$\mathcal{USC}^{\mathcal{I}}$ is the set of upper semicontinuous, locally bounded functions on $\mathbb{R}^D \times [0, T]$ such that $M(u(x+z, t), u(x, t))$ has an upper μ -bound at any (x, t) ;
 $\mathcal{LSC}^{\mathcal{I}}$ is the set of lower semicontinuous, locally bounded functions on $\mathbb{R}^D \times [0, T]$ such that $M(u(x+z, t), u(x, t))$ has a lower μ -bound at any (x, t) ;
 $\mathcal{C}^{\mathcal{I}} = \mathcal{USC}^{\mathcal{I}} \cap \mathcal{LSC}^{\mathcal{I}}$.

It can be observed that if the integral operator \mathcal{I} has some more regularities, the admissible classes described before coincide with the classes of exponential growth at infinity, that are the classes of growth required in the framework of pure diffusion models; for details we refer to [3,4].

Finally we can define viscosity sub/super solutions in the integro-differential framework.

Definition 3.3. $u \in \mathcal{USC}^{\mathcal{I}}$ ($u \in \mathcal{LSC}^{\mathcal{I}}$) is a **viscosity subsolution** (resp. **viscosity supersolution**) of the equation (3.1) if:

$$\partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2 u) \leq 0 \quad (\text{resp. } \geq)$$

holds in viscosity sense for all $(x, t) \in \mathbb{R}^D \times (0, T)$. If in addition:

$$u(x, 0) \leq u_0(x) \quad (\text{resp. } \geq)$$

for all $x \in \mathbb{R}^D$, then u is a viscosity subsolution (resp. viscosity supersolution) of the integro-differential Cauchy problem (3.1). An arbitrary function $u : \mathbb{R}^D \times [0, T) \rightarrow \mathbb{R}$ is a **weak viscosity solution** for the problem (3.1) if its upper/lower semicontinuous envelopes belong to $\mathcal{USC}^{\mathcal{I}}/\mathcal{LSC}^{\mathcal{I}}$, respectively, and they are sub/supersolutions. Besides, it is a (strong) **viscosity solution** for the problem (3.1) if it is both a sub/super solution.

In this framework, using Perron method and comparison principles and with some further assumptions, it is possible to give a result of existence, uniqueness and regularity for the solutions of the integro-differential Cauchy problem associated to problem (2.4). For more precise statements and proofs we refer to [3]. In the following statement the index *pol* stands for a polynomial growth of the considered norms at infinity.

Theorem 3.2. Assume that the parameters α, σ, r satisfy some proper regularity conditions of continuity and Lipschitz continuity and the market is without arbitrage opportunities; then, for every final value $G \in \mathcal{C}((0, +\infty)^D) \cap W_{pol}^{1,\infty}((0, +\infty)^D)$, the integro-differential Cauchy problem (2.4) has an unique viscosity solution V , in the sense of Definition 3.3, which belongs to $L^\infty(0, T; W_{pol}^{1,\infty}((0, +\infty)^D))$. Moreover, comparison principle applies and we have that $V \geq 0$ whenever $G \geq 0$.

4. A general convergence result

We define a numerical grid in $\mathbb{R}^D \times (0, T)$ using the following notation: $h = (h_1, \dots, h_D)$ is the spatial grid size, k is the time grid size, $(x_j, t_n) = (jh, nk)$, $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, are the grid points, v_j^n is the value of the function v , defined on the grid or defined for continuously varying (x, t) , at the grid point (x_j, t_n) and \tilde{v} is the vector of v values, $(v_j^n)_j$ for j varying on a subset of \mathbb{Z} and $n \in \mathbb{N}$.

We want to approximate the following problem:

$$(4.1) \quad \partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2 u) = 0.$$

A numerical scheme approximating (4.1) can be written as

$$(4.2) \quad Q(h, k, j, n, u_j^n, \mathcal{I}_h \tilde{u}, \tilde{u}) = 0,$$

where $\mathcal{I}_h \tilde{u}$ denotes the integral approximation. We want to prove that, under suitable conditions, this scheme converges to the solution of the problem (4.1), provided that this problem satisfies proper conditions.

Properties of the scheme

H1 Monotonicity of the approximating integral.

If $\tilde{u} \geq \tilde{v}$ and $u_j^n = v_j^n$ we have the following inequality:

$$(4.3) \quad \mathcal{I}_h \tilde{u} \geq \mathcal{I}_h \tilde{v};$$

H2 Stability.

For all h, k a solution \hat{u} does exist that is bounded

$$(4.4) \quad \text{independently from } (h, k);$$

H3 Consistency.

For all $\phi \in C_b^\infty(\mathbb{R}^D \times [0, T])$ and for all $(x, t) \in \mathbb{R}^D \times (0, T)$ we have:

$$(4.5) \quad \liminf_{\substack{(h,k) \rightarrow 0 \\ (j,h,nk) \rightarrow (x,t) \\ \xi \rightarrow 0}} \frac{Q(h, k, j, n, \phi_j^n + \xi, \mathcal{I}_h(\tilde{\phi} + \xi), \tilde{\phi} + \xi)}{\rho(h, k)} \geq \partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2 u);$$

$$(4.6) \quad \limsup_{\substack{(h,k) \rightarrow 0 \\ (j,h,nk) \rightarrow (t,x) \\ \xi \rightarrow 0}} \frac{Q(h, k, j, n, \phi_j^n + \xi, \mathcal{I}_h(\tilde{\phi} + \xi), \tilde{\phi} + \xi)}{\rho(h, k)} \leq \partial_t u + F(x, t, u, \mathcal{I}u, \mathcal{D}u, \mathcal{D}^2 u);$$

H4 Monotonicity.

If $\tilde{u} \geq \tilde{v}$ and $u_j^n = v_j^n$ for all $h, k \geq 0$ and $1 \leq n \leq N$, we have:

$$(4.7) \quad Q(h, k, n, j, u_j^n, \mathcal{I}_h \tilde{u}, \tilde{u}) \leq Q(h, k, n, j, v_j^n, \mathcal{I}_h \tilde{v}, \tilde{v}).$$

Remark 4.1. The theory of numerical approximation of fully nonlinear degenerate parabolic problems [9] could be considered as a special case of the present one. We define the numerical scheme approximating the parabolic problem:

$$(4.8) \quad \partial_t u + F(x, t, u, 0, \mathcal{D}u, \mathcal{D}^2 u) = 0 \text{ in } \mathbb{R}^D \times (0, T),$$

as:

$$(4.9) \quad \tilde{Q}(h, k, j, n, u_j^n, \tilde{u}) = Q(h, k, n, j, u_j^n, 0, \tilde{u});$$

in this way the scheme \tilde{Q} satisfies clearly all the properties required by Barles and Souganidis in [9], and therefore the approximation scheme (4.9) converges to the viscosity solutions of (4.8).

Properties of the equation

H5 Maximum Principle or Strong Uniqueness Property Let $u_0 \in \mathcal{C}(\mathbb{R}^D) \cap L_{\text{exp}}^\infty(\mathbb{R}^D)$ be the initial data of (4.1), (1.1)₂, such that there exists $m > 0$:

$$|u_0(x)| \leq B e^{n_0 \|x\|} \text{ for } n_0 \leq m;$$

if $u \in \bigcup_{n < m} L^\infty(0, T; L_{e^{\|\cdot\|}}^\infty(\mathbb{R}^D))$ is an $\mathcal{USC}^{\mathcal{I}}$ subsolution of (4.1) and $v \in \bigcup_{n < m} L^\infty(0, T; L_{e^{\|\cdot\|}}^\infty(\mathbb{R}^D))$ is a $\mathcal{LSC}^{\mathcal{I}}$ supersolution of (4.1), then

$$u \leq v \text{ on } \mathbb{R}^D \times [0, T].$$

Under these assumptions we shall prove our main theoretical result.

Theorem 4.2. *Let assumption (H1)–(H5) hold true. Then, as $(h, k) \rightarrow 0$, the solution \tilde{u} of the scheme (4.2) converges locally uniformly to the unique continuous viscosity solution of the problem (4.1)*

Proof. Let $\underline{u}, \bar{u} \in \bigcup_{n < m} L^\infty(0, T; L_{e^{\|\cdot\|}}^\infty(\mathbb{R}^D))$ be defined by:

$$\begin{aligned} \underline{u}(x, t) &= \liminf_{\substack{(\Delta t, \Delta x) \rightarrow 0 \\ (n \Delta t, j \Delta x) \rightarrow (t, x)}} u_j^n, \\ \bar{u}(x, t) &= \limsup_{\substack{(\Delta t, \Delta x) \rightarrow 0 \\ (n \Delta t, j \Delta x) \rightarrow (t, x)}} u_j^n. \end{aligned}$$

We want to prove that \underline{u} and \bar{u} are respectively supersolution and subsolution of the problem (4.1). If this claim is proved to be true, then by definition we have $\underline{u} \leq \bar{u}$, while the other inequality holds because of lower semicontinuity of \underline{u} and upper semicontinuity of \bar{u} , hence:

$$u = \underline{u} = \bar{u},$$

is the unique continuous solution of the problem (4.1). This result, together with the definition of \underline{u} and of \bar{u} leads to the local uniform convergence of

the solution of the scheme to the solution of the problem. To prove the claim, we will consider only the case of \bar{u} , the other being the same. We want to prove that $\bar{u} \in \mathcal{USC}^{\mathcal{I}}$ and that it is a subsolution for the problem, i.e. for all $\phi \in \mathcal{C}^{2,1}(\mathbb{R}^D \times [0, T])$ such that $\bar{u} - \phi$ has a local maximum in (x_0, t_0) we have:

$$\partial_t \phi(x_0, t_0) + F(x_0, t_0, \bar{u}(x_0, t_0), \mathcal{I}\phi(x_0, t_0), \mathcal{D}\phi(x_0, t_0), \mathcal{D}^2\phi(x_0, t_0)) \leq 0.$$

We start by proving that $\bar{u} \in \mathcal{USC}^{\mathcal{I}}$.

- (1) \bar{u} is upper semicontinuous: it follows from the definition itself: we want to prove that the following inequality holds:

$$\limsup_{(y,s) \rightarrow (x,t)} \bar{u}(y, s) \leq \bar{u}(x, t),$$

By definition:

$$\bar{u}(y, s) = \limsup_{\substack{(\Delta t, \Delta x) \rightarrow 0 \\ (n\Delta t, j\Delta x) \rightarrow (s, y)}} u_j^n,$$

therefore, by definition of limsup, there exist an $\varepsilon > 0$ and (n, j) such that:

$$\bar{u}(y, s) - \varepsilon \leq u_j^n;$$

now, taking limsup for $(\Delta t, \Delta x) \rightarrow 0$ and $(n\Delta t, j\Delta x) \rightarrow (t, x)$, we obtain:

$$\bar{u}(y, s) - \varepsilon \leq \bar{u}(x, t);$$

Now, as ε is arbitrarily chosen, we obtain the desired result.

- (2) \bar{u} is locally bounded: by definition it is obtained by the solutions of the scheme. By hypothesis (4.4) they are bounded, independently from $\Delta t, \Delta x$. So let $K \in \mathbb{R}^D \times [0, T]$ be a compact set, then there exists a constant A_K such that:

$$\begin{aligned} |u_j^n| &\leq A_K \quad \forall n, j \text{ s.t. } (n\Delta t, j\Delta x) \in K \Rightarrow |\bar{u}(x, t)| \\ &\leq A_K \quad \forall (x, t) \in K; \end{aligned}$$

- (3) $M(\bar{u}(x+z, t), \bar{u}(x, t))$ has an upper μ -bound in (x, t) . From the hypotheses on M , it clearly follows that M is a Lipschitz function with constant c , so we have:

$$M(\bar{u}(x+z, t), \bar{u}(x, t)) \leq c|\bar{u}(x+z, t) - \bar{u}(x, t)|;$$

it is easily shown that in a compact neighborhood $V_{x,t}$ of (x, t) we have:

$$M(\bar{u}(x+z, t), \bar{u}(x, t)) \leq 2cA_{V_{x,t}}.$$

It is then sufficient to choose Φ as a constant to be the wanted μ -bound.

We have proved that $\bar{u} \in \mathcal{USC}^{\mathcal{I}}$, and now we need only to prove that \bar{u} is a viscosity subsolution. To this aim let (x_0, t_0) be a global strict maximum for $\bar{u} - \phi$ on $\mathbb{R}^D \times [0, T]$ for some $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^D \times [0, T])$. We could assume that $\bar{u}(x_0, t_0) = \phi(x_0, t_0)$ and that :

$$\bar{u}(x, t) - \phi(x, t) \leq 0 = \bar{u}(x_0, t_0) - \phi(x_0, t_0) \text{ in } \mathbb{R}^D \times [0, T].$$

From these hypothesis it follows that there exists a sequence $(\Delta t_k, \Delta x_k) \in \mathbb{R}^{+2}$ and $(y_k, s_k) \in \mathbb{R}^D \times [0, T]$ such that, as $k \rightarrow \infty$:

$$(4.10) \quad \begin{aligned} & (\Delta t_k, \Delta x_k) \rightarrow 0, \quad (y_k, s_k) \rightarrow (x_0, t_0), \quad u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) \rightarrow \bar{u}(x_0, t_0), \\ & (y_k, s_k) \text{ is a global maximum point of } u^{(\Delta t_k, \Delta x_k)}(\cdot, \cdot) - \phi(\cdot, \cdot). \end{aligned}$$

Denoting by $\xi_k = u^{(\Delta t_k, \Delta x_k)}(y_k, s_k) - \phi(y_k, s_k)$, we have

$$\begin{aligned} & \xi_k \rightarrow 0 \text{ and} \\ & u^{(\Delta t_k, \Delta x_k)}(x, t) \leq \phi(x, t) + \xi_k, \end{aligned}$$

for all $(x, t) \in \mathbb{R}^D \times [0, T]$. By the definition of $u^{(\Delta t_k, \Delta x_k)}$, the hypotheses (4.7), (4.10) and the hypotheses on M we obtain:

$$Q(\Delta t_k, \Delta x_k, n_k, j_k, \phi(y_k, s_k) + \xi_k, \mathcal{I}_{n_k, j_k}(\tilde{\phi} + \xi_k), \tilde{\phi} + \xi_k) \leq 0.$$

Now, taking limits in the previous inequality, using the consistency of the scheme, we obtain:

$$\begin{aligned} 0 & \geq \liminf_{k \rightarrow \infty} \frac{Q(\Delta t_k, \Delta x_k, n_k, j_k, \phi(y_k, s_k) + \xi_k, \mathcal{I}_{n_k, j_k}(\tilde{\phi} + \xi_k), \tilde{\phi} + \xi_k)}{\rho_k(\Delta t, \Delta x)} \\ & \geq \liminf_{\substack{(\Delta t, \Delta x) \rightarrow 0 \\ (n\Delta t, j\Delta x) \rightarrow (t, x) \\ \xi \rightarrow 0}} \frac{Q(\Delta t, \Delta x, n, j, \phi(y, s) + \xi, \mathcal{I}(\tilde{\phi} + \xi), \tilde{\phi} + \xi)}{\rho(\Delta t, \Delta x)} \\ & \geq \partial_t \phi + F(x, t, \phi, \mathcal{I}\phi, \mathcal{D}\phi, \mathcal{D}^2\phi), \end{aligned}$$

which is the desired result, because of the assumption $\bar{u}(x_0, t_0) = \phi(x_0, t_0)$. \square

5. The numerical approximation of the integral term

According to the classical theory of approximated integration, see for instance [19], we use the compound Newton-Cotes formulas to approximate the integral term on the interval $[a, b]$:

$$(5.1) \quad (R_S)(f) = \frac{b-a}{2S} \sum_{s=0}^{S-1} \sum_{i=1}^{\rho} \alpha_i f(x_{is}) \approx \int_a^b f(x) dx,$$

where S is the number of subinterval in which we have divided $[a, b]$, $a = y_0 < y_1 < \dots < y_S = b$, ρ is the number of point in each subinterval $[y_s, y_{s+1}]$ and

$$x_{is} = y_s + \frac{b-a}{2S}(1+t_i), \quad s = 0, \dots, S-1.$$

The errors which occur in approximate integration formulas are conventionally expressed in terms of the higher derivatives of the integrand function f and they are valid only if the integrand is sufficiently smooth. It is a feature of the Newton-Cotes formulas that, if the number of point is $2k-1$ or $2k$, the error is of the form $\mathcal{E}_R(f) = ch^{2k+1}f^{2k}(\xi)$, for $a < \xi < b$. To obtain the error estimates when the integrand function f has a low-order continuity, we can approximate the function f by a suitable polynomial, according to the following standard result, see for instance [19].

Proposition 5.1. *Let $f(x)$ be of class $\mathcal{C}[a, b]$, and let $\omega(\delta)$ be its modulus of continuity. Then for each $n \in \mathbb{N}$, there exists a polynomial of degree $\leq n$, $p_n(x)$, such that*

$$|f(x) - p_n(x)| \leq 2\omega\left(\frac{b-a}{2n}\right), \quad a \leq x \leq b.$$

If $f(x)$ is of class $\mathcal{C}[a, b]$ and has a bounded derivative,

$$|f'(x)| \leq M, \quad a \leq x \leq b,$$

then for each $n \in \mathbb{N}$ there exists a polynomial $p_n(x)$ of degree $\leq n$ such that

$$|f(x) - p_n(x)| \leq \frac{3(b-a)M}{n}, \quad a \leq x \leq b.$$

5.1. The numerical approximation in the one dimensional case

Here we want to consider the integro-differential equation (3.1) in one dimension. We suppose F to be linear in the integral part,

$$(5.2) \quad \partial_t u + F(x, t, u, \mathcal{D}u, \mathcal{D}^2 u) - \mathcal{I}u = 0.$$

The first step to approximate the integral operator (1.2) using the numerical integrations formula described in the previous section is to truncate the integral domain; let us choose the interval $[z_m, z_M]$ such that

$$(5.3) \quad \int_{z_m}^{z_M} \mu_{x,t}(dz) \approx \int_{-\infty}^{+\infty} \mu_{x,t}(dz) - \nu = 1 - \nu, \quad \nu \ll 1.$$

Assume $u(\cdot, t) \in L^\infty(\mathbb{R})$ and let U be its L^∞ -norm. The error due to the truncation of the domain is estimated as follows.

$$\begin{aligned}
& \left| \int_{-\infty}^{+\infty} M(u(x+z, t), u(x, t)) \mu_{x,t}(dz) \right. \\
& \quad \left. - \int_{z_m}^{z_M} M(u(x+z, t), u(x, t)) \mu_{x,t}(dz) \right| \\
&= \left| \int_{-\infty}^{z_m} M(u(x+z, t), u(x, t)) \mu_{x,t}(dz) \right| \\
& \quad + \left| \int_{z_M}^{+\infty} M(u(x+z, t), u(x, t)) \mu_{x,t}(dz) \right| \\
&\leq c \int_{-\infty}^{z_m} |u(x+z, t) - u(x, t)| \mu_{x,t}(dz) \\
& \quad + c \int_{z_M}^{+\infty} |u(x+z, t) - u(x, t)| \mu_{x,t}(dz) \\
&\leq 2Uc \left[\int_{-\infty}^{z_m} \mu_{x,t}(dz) + \int_{z_M}^{+\infty} \mu_{x,t}(dz) \right] \\
(5.4) \quad &= 2Uc \int_{\mathbb{R}-[z_m, z_M]} \mu_{x,t}(dz) = 2Uc v.
\end{aligned}$$

5.2. The case of the Gaussian distribution

In the estimation (5.4), we have supposed that the function $u(\cdot, t)$ is $L^\infty(\mathbb{R})$, but it is possible to suppose even more regularity, under particular hypothesis on the Radon measure. Let us consider, for example, on the tracks of Merton, the Gaussian distribution. It leads to a probability measure with the property of being symmetric with respect to the origin of the real line. In this case it is possible to assume that $u(\cdot, t) \in Lip(\mathbb{R})$ with constant L ; we remember that in this particular case, the integral term is:

$$\mathcal{I}u = \lambda \int_{-\infty}^{+\infty} [u(x+z, t) - u(x, t)] \mu_{x,t}(z) dz,$$

where $\mu_{x,t} = \Gamma_\delta$ is the Gaussian probability density:

$$\mu_{x,t}(dz) := \Gamma_\delta(z) dz = \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{z^2}{2\delta^2}\right) dz.$$

As was previously shown, the calculation of the integral term could be simplified by considering a finite interval instead of the whole real line. Thanks to the particular shape of the density measure Γ_δ , we can select the finite interval considering only those points for which the density has a significant

value and this choice would not introduce big errors. Choose a parameter $\varepsilon > 0$ and select the interval $[z_m, z_M]$ as the set of all the points z that verify:

$$\Gamma_\delta(z) \geq \varepsilon \iff \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{z^2}{2\delta^2}} \geq \varepsilon;$$

by simple calculation we can derive z_m and z_M :

$$-\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})} \leq z \leq \sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}.$$

As Γ_δ is a symmetric function with respect to its axis (that in this case is the line $z = 0$), we define:

$$z_M = \sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}, \quad z_m = -z_M.$$

Under these hypotheses we have the following estimate:

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} M(u(x+z, t), u(x, t)) \Gamma_\delta(dz) \right. \\ & \quad \left. - \int_{z_m}^{z_M} M(u(x+z, t), u(z, t)) \Gamma_\delta(dz) \right| \\ & \leq L \left(\int_{-\infty}^{z_m} |z| \Gamma_\delta(dz) + \int_{z_M}^{+\infty} |z| \Gamma_\delta(dz) \right), \end{aligned}$$

therefore

$$\begin{aligned} & \int_{-\infty}^{z_m} |z| \Gamma_\delta(dz) + \int_{z_M}^{+\infty} |z| \Gamma_\delta(dz) \\ (5.5) \quad & = 2 \int_{z_M}^{+\infty} z \frac{1}{\sqrt{2\pi}\delta} \exp\left(-\frac{z^2}{2\delta^2}\right) dz = \frac{2\delta^2}{\sqrt{2\pi}} \exp\left(-\frac{z_M^2}{2\delta^2}\right) = 2\delta^2 \varepsilon. \end{aligned}$$

Let us now apply the compound rule (5.1) to the truncated integral.

$$\begin{aligned} \mathcal{I}_h u &= \lambda(R_S)(M\Gamma_\delta) = \lambda \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^{\rho} \alpha_i \\ (5.6) \quad & \times M(u(x+z_{is}, t), u(x, t)) \Gamma_\delta(z_{is}). \end{aligned}$$

Since the function $g(z) = M(u(x+z, t), u(x, t)) \Gamma_\delta(z)$ has a low-order continuity, to get an error estimate for the approximation (5.6), we apply Proposition 5.1. Recall that for a generic function f we have

$$\begin{aligned} \mathcal{E}_R(f) &= \int_a^b f(x) dx - R(f) = \int_a^b (f(x) - p_n(x)) dx \\ & \quad + \int_a^b p_n(x) dx - R(f) \\ &= \int_a^b (f(x) - p_n(x)) dx + R(p_n - f). \end{aligned}$$

Then

$$| \mathcal{E}_R(f) | \leq \left((b-a) + \sum_{i=0}^{\rho} | \alpha_i | \right) | f(x) - p_n(x) | .$$

If $\alpha_i > 0$, using formula (5.1), we obtain

$$| \mathcal{E}_R(f) | \leq 2(b-a) | f(x) - p_n(x) | .$$

An (R_S) compound rule, applied to our function g , yields

$$\mathcal{E}_{R_S}(g) = \sum_{s=0}^{S-1} \mathcal{E}_R(g) = \sum_{s=0}^{S-1} 2 \left(\frac{b-a}{S} \right) | g(x_s) - p_n(x_s) | .$$

Then there exists a polynomial of degree $\leq S\rho$, $p_{S\rho}(z)$, such that

$$| g(z) - p_{S\rho}(z) | \leq 2\omega \left(\frac{z_M - z_m}{2S\rho} \right) .$$

There follows

$$\begin{aligned} & \left| \int_{z_m}^{z_M} g(z) dz - (R_S)(g(z)) \right| \\ & \leq \int_{z_m}^{z_M} | g(z) - p_{S\rho}(z) | dz + \left| (R_S)(p_{S\rho}(z) - g(z)) \right| \\ & \leq \left((z_M - z_m) + \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^{\rho} | \alpha_i | \right) 2\omega \left(\frac{z_M - z_m}{2S\rho} \right) \\ (5.7) \quad & \leq 2(z_M - z_m) \left(1 + \frac{1}{2} \sum_{i=0}^{\rho} | \alpha_i | \right) \omega \left(\frac{h}{2} \right), \end{aligned}$$

where ω is the modulus of continuity for g .

5.3. Check of the hypotheses of Theorem 4.2 for the integral part

First, we have to approximate the differential operator $\partial_t + F$: we take a numerical scheme \tilde{Q} that verifies the convergence (differential) conditions (H_2) - (H_4) of [9]. In particular, to keep the order of the convergence of the integration formula (5.6), we assume that the space discretization grid of the numerical operator \tilde{Q} coincides with the integral one, i.e. we set the common space step h such that

$$h \leq \frac{z_M - z_m}{p \cdot S} .$$

Then the approximation of the integro-differential equation (5.2) is given by:

$$Q(h, k, j, n, u_j^n, \mathcal{I}_h \tilde{u}, \tilde{u}) = \tilde{Q}(h, k, j, n, u_j^n, \tilde{u}) - \mathcal{I}_h \tilde{u} = 0,$$

We want to show that under the above assumption, this scheme satisfies conditions (4.3)-(4.7).

1. *Monotonicity of the approximating integral*

Since the function M is such that

$$M(u, w) \leq M(v, w), \text{ if } u \leq v,$$

to get the monotonicity of the integral approximation it is sufficient that the weights α_i are greater than zero for all i . Clearly, if $\tilde{u} \leq \tilde{v}$ and $u_j^n = v_j^n$, we have

$$\begin{aligned} & \lambda \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^p \alpha_i M(u^n(x_j + z_{is}), u_j^n) \Gamma_\delta(z_{is}) \\ & \leq \lambda \frac{z_M - z_m}{2S} \sum_{s=0}^{S-1} \sum_{i=0}^p \alpha_i M(v^n(x_j + z_{is}), v_j^n) \Gamma_\delta(z_{is}), \end{aligned}$$

for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$.

2. *Stability*

It is a trivial consequence of the \tilde{Q} stability (4.4) and the monotonicity of the integral approximation.

3. *Consistency*

Let $\phi \in C^\infty(\mathbb{R} \times (0, T))$, from the consistency condition (4.5) on \tilde{Q} , we get the following inequality:

$$\begin{aligned} & \liminf_{\substack{(h,k) \rightarrow 0 \\ (jh,nk) \rightarrow (x,t) \\ \xi \rightarrow 0}} \frac{\tilde{Q}(h, k, j, n, \phi_j^n + \xi, \tilde{\phi} + \xi) - \mathcal{I}_h(\tilde{\phi} + \xi)}{\rho(h, k)} \\ & \geq \partial_t u + F(x, t, u, \mathcal{D}u, \mathcal{D}^2u) - \liminf_{\substack{(h,k) \rightarrow 0 \\ (jh,nk) \rightarrow (x,t) \\ \xi \rightarrow 0}} \frac{\mathcal{I}_h(\tilde{\phi} + \xi)}{\rho(h, k)}. \end{aligned}$$

From the error estimate of the integral approximation (5.7), we have

$$\liminf_{\substack{(h,k) \rightarrow 0 \\ (jh,nk) \rightarrow (x,t) \\ \xi \rightarrow 0}} \frac{\mathcal{I}_h(\phi + \xi)}{\rho(h, k)} = \mathcal{I}\phi - \lim_{(h,k) \rightarrow 0} \frac{\mathcal{E}_{R_S}(\phi + \xi)}{\rho(h, k)} = \mathcal{I}\phi.$$

then, we get condition (4.5). Condition (4.6) follows by analogous considerations.

4. *Monotonicity*

It is a trivial consequence of the \tilde{Q} monotonicity (4.7) and the monotonicity of the integral approximation (point 1).

Remark 5.2. In our numerical test, we have always considered a Radon measure absolutely continuous with respect to the Lebesgue measure, i.e:

$$\mu_{x,t}(dz) = \lambda \Gamma_\delta(z) dz.$$

It is even possible to consider a discrete measure, for example the Dirac measure:

$$\mu_{x,t}(dz) = \delta_{z_0}(z) dz.$$

In that case the numerical approximation is even simpler, thanks to the absence of the integral term.

6. The diffusive effect of the integral operator

An important point in the numerical simulation for the problem we have presented, is the behaviour of the solution at the limiting point of the truncated numerical domain. In this particular framework, the presence of the integral term which convolutes "internal" and "external" points requires a particular tool to deal with such a difficulty. One possibility is to look at the particular form of the integral term $\mathcal{I}u$ with respect to the Gaussian parameter δ : we show that a convenient way to deal with the integral operator is to replace it (locally) by an effective diffusion term. This result will be useful in the numerical simulations, as is shown next in Subsection 7.2. The following discussion, which is presented only in the linear case, has the main purpose of rigorously investigating the error generated by this approximation. Let us consider the two following one dimensional equations, for $(x, t) \in \mathbb{R} \times (0, T)$:

$$(6.1) \quad u_t + au_x - bu_{xx} + cu = \mathcal{I}u,$$

$$(6.2) \quad v_t + av_x - bv_{xx} + cv = \frac{\lambda\delta^2}{2} v_{xx},$$

with the same initial condition

$$u(x, 0) = v(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$

It is possible to prove that, under proper hypotheses on the density distribution Γ_δ and on the solutions u and v , the integral problem (6.1) is well approximated by the advection-diffusion one (6.2).

Proposition 6.1. *Let u be the solution of problem (6.1) and v the solution of problem (6.2) with the same initial condition $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, if $\delta \ll 1$, there holds*

$$\|u - v\|_{L^\infty(0,T;L^1(\mathbb{R}))} \leq O(T\delta^3).$$

Proof. The function $w = u - v$ is a solution of the following problem (written in the weak formulation):

$$\begin{aligned}
& - \int_0^T \int_{-\infty}^{+\infty} \left[\phi_t(x, t) + a\phi_x(x, t) + b\phi_{xx}(x, t) - c\phi(x, t) \right] w(x, t) dx dt \\
& = \lambda \int_0^T \int_{-\infty}^{+\infty} \frac{\delta^2}{2} \phi_{xx}(x, t) w(x, t) dx dt \\
& \quad + \lambda \int_0^T \int_{-\infty}^{+\infty} u(x, t) \int_{-\infty}^{+\infty} \left[\phi(x+z, t) - \phi(x, t) \right. \\
(6.3) \quad & \left. - \frac{\delta^2}{2} \phi_{xx}(x, t) \right] \Gamma_\delta(z) dz dx dt,
\end{aligned}$$

for every test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R} \times [0, T])$.

To estimate the inner integral in the second member of the RHS we can take the Taylor expansion of ϕ , which leads to

$$\begin{aligned}
& \phi(x+z, t) - \phi(x, t) - \frac{\delta^2}{2} \phi_{xx}(x, t) \\
& = z\phi_x(x, t) + \frac{z^2 - \delta^2}{2} \phi_{xx}(x, t) \\
& \quad + \frac{z^3}{6} \int_0^1 (1-k)^3 \phi_{xxx}(x + (1-k)z, t) dk.
\end{aligned}$$

We can estimate in term of the norm of u , the error made by using this expansion:

$$\begin{aligned}
& \left| \lambda \int_0^T \int_{-\infty}^{+\infty} u(x, t) \int_{-\infty}^{+\infty} \left[\phi(x+z, t) - \phi(x, t) \right. \right. \\
& \quad \left. \left. - \frac{z^2}{2} \phi_{xx}(x, t) \right] \Gamma_\delta(z) dz dx dt \right| \\
& \leq \lambda \left| \int_0^T \int_{-\infty}^{+\infty} u(x, t) \int_{-\infty}^{+\infty} \frac{z^3}{6} \left[\int_0^1 (1-k)^3 \phi_{xxx} \right. \right. \\
& \quad \left. \left. \times (x + (1-k)z, t) dk \right] \Gamma_\delta(z) dz dx dt \right| \\
& \leq \int_0^T \|\phi(\cdot, t)\|_{\mathcal{C}_0^3(\mathbb{R})} \|u(\cdot, t)\|_{L^1(\mathbb{R})} dt \int_{-\infty}^{+\infty} \frac{|z|^3}{6} \Gamma_\delta(z) dz \\
& \leq \lambda \frac{\delta^3}{\sqrt{2\pi}} \|\phi\|_{\mathcal{C}_0^3(\mathbb{R} \times [0, T])} T \|u\|_{L^\infty(0, T; L^1(\mathbb{R}))} = O(T\delta^3).
\end{aligned}$$

The RHS of (6.3) can be rewritten taking into account the last estimate:

$$\begin{aligned}
& - \int_0^T \int_{-\infty}^{+\infty} \left[\phi_t(x, t) + a\phi_x(x, t) + b\phi_{xx}(x, t) - c\phi(x, t) \right] w(x, t) dx dt \\
& = \lambda \int_0^T \int_{-\infty}^{+\infty} \frac{\delta^2}{2} \phi_{xx} w(x, t) dx dt \\
& \quad + \lambda \int_0^T \int_{-\infty}^{+\infty} w(x, t) \int_{-\infty}^{+\infty} \left[\phi_x(x, t)z + \frac{z^2 - \delta^2}{2} \phi_{xx}(x, t) \right] \\
& \quad \times \Gamma_\delta(z) dz dx dt + O(T\delta^3).
\end{aligned}$$

Since for the inner integral, there holds

$$\int_{-\infty}^{+\infty} \left[\phi_x(x, t)z + \frac{z^2 - \delta^2}{2} \phi_{xx}(x, t) \right] \Gamma_\delta(z) dz = 0,$$

w is just the weak solution to problem

$$w_t + aw_x - \left(b + \lambda \frac{\delta^2}{2} \right) w_{xx} + cw = O(T\delta^3),$$

with initial datum

$$w(x, 0) = 0, \quad x \in \mathbb{R},$$

which yields

$$\|w\|_{L^\infty(0, T; L^1_{loc}(\mathbb{R}))} \leq O(T\delta^3).$$

Now, taking a suitable sequence of test functions such that $\overline{\text{supp } \phi(x)} = [-R, R]$, and letting $R \rightarrow +\infty$, gives the result. \square

7. Finite difference methods for the one dimensional jump-diffusion model

In this section we introduce an explicit approximation for the linear PIDE arising from the jump-diffusion models and we give a convenient way to deal with the problem of the numerical boundary conditions.

First of all it is important to recall that a huge literature exists for the pure diffusion Black-Scholes problem (2.1) within the subject of numerical approximation for the linear convection-diffusion equations. Let us shortly introduce a standard explicit 3-points finite-difference scheme for the Black-Scholes equation

$$(7.1) \quad u_t + \mathcal{L}u = u_t - bu_{xx} + au_x + cu = 0,$$

where $a = -(r - \sigma^2/2)$, $b = \sigma^2/2$ and $c = r > 0$, is

$$(7.2) \quad \begin{aligned} \tilde{Q}(h, k, j, n, u_j^n, \tilde{u}) &= \frac{u_j^{n+1} - u_j^n}{k} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} \\ &- \left(\frac{q}{2k} + \frac{b}{h^2}\right)(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + cu_j^n = 0. \end{aligned}$$

The q parameter is connected with the numerical viscosity of the scheme. In order to verify the monotonicity and stability hypotheses, the scheme has to satisfy the following Courant-Friedrichs-Levy (CFL) condition

$$\frac{|a|k}{h} \leq \frac{2bk}{h^2} + q \leq 1 - ck.$$

Usual values of q are given by: $q = 0$, the standard central scheme, which is second order, but stable only under the CFL condition $k \leq \min(\frac{2b}{a^2+2bc}, \frac{h^2}{2b+ch^2})$; $q = \frac{|a|k}{h}$, the upwind scheme, which is first order, but stable for $k \leq \frac{h^2}{2b+|a|h+ch^2}$.

The most elementary way to avoid the CFL conditions is to use an implicit scheme in time, such as a Crank-Nicholson scheme, given by

$$(7.3) \quad \tilde{Q}(h, k, j, n, u_j^n, \tilde{u}) = \frac{u_j^{n+1} - u_j^n}{k} + \mathcal{L}[\theta u_j^n + (1 - \theta)u_j^{n+1}] = 0.$$

with $\theta = 1/2$.

We extend now the discussion to the PIDE. After appropriate logarithmic transformations the Merton problem (2.3) becomes

$$(7.4) \quad \begin{cases} u_t + au_x = bu_{xx} - cu + \lambda \left(\int_{-\infty}^{\infty} u(x+z, t) \Gamma_{\delta}(z) dz - u \right), \\ u(x, 0) = \psi(x), \end{cases}$$

where

$$a = -(r - \lambda \bar{k} - \frac{1}{2}\sigma^2), \quad b = \frac{1}{2}\sigma^2, \quad c = r, \quad \bar{k} = \mathbb{E}(\eta - 1),$$

and the initial data $\psi(x)$ is the payoff function of the European contingent claim. Let the exercise price E be given, we have

$$\psi(x) = (e^x - E)_+ \text{ and } \psi(x) = (E - e^x)_+,$$

for the call and the put option respectively.

As done in (7.3), we can write the time approximation of the PIDE (7.4) in the following “ θ -form”:

$$(7.5) \quad \frac{u_j^{n+1} - u_j^n}{k} + \mathcal{L}[\theta_1 u_j^n + (1 - \theta_1)u_j^{n+1}] + \theta_2 \mathcal{I}u_j^n + (1 - \theta_2)\mathcal{I}u_j^{n+1} = 0,$$

where $\theta_1, \theta_2 \in [0, 1]$. The choice $\theta_1 = \theta_2 = 0$ gives the explicit scheme, while $\theta_1 = \theta_2 = 1$ gives an implicit time differencing scheme, unconditionally stable, but not practically feasible. Actually the convolution integral introduces a significant complication for the numerical solution, since it couples grid points over an extended range, leading to a dense system of equations which is hard to be solved. In fact, after discretizing the x -space into N points the inversion of a full $N \times N$ matrices is required. For $\theta_1 = 1/2, \theta_2 = 0$ it gives an asymmetric treatment (implicit-explicit) of the differential and integral part. This is a way to avoid dense systems, but it is only first order in time.

In the book [35], Tavella and Randall propose an iterative approach to avoid dense systems and to increase the convergence order in time. They write the time-discretized equation as

$$\frac{u^{m+1} - u^n}{k} + \mathcal{L} \frac{u^{m+1} + u^n}{2} - \lambda \left(\int_{-\infty}^{\infty} \frac{u^m(x+z) + u^n(x+z)}{2} \Gamma_{\delta}(z) dz - \frac{u^m + u^n}{2} \right) = 0.$$

At each time step, the iteration begins with $u^m = u^n$, then proceeds by solving for u^{m+1} and substituting the new u^{m+1} for u^m . The iteration proceeds until a convergence criterion is met. Here they set $u^{m+1} \approx u^{n+1}$ and a new time step begins.

Due to the iteration procedure, this method turns out to be computationally heavy and it is still not clear how to select a good stop criterion.

In the article [6], Andersen and Andreasen proposed an FFT-ADI (Fast Fourier Transform - Alternating Directions Implicit) to avoid the conditional stability of explicit methods. The FFT technique is applied to the convolution integral and coupled with an ADI method where each time step is split into two half steps: the idea is to choose in the time approximation (7.5), $\theta_1 = 1$ and $\theta_2 = 0$ for the half time step $t_n \rightarrow t_{n+1/2}$ and $\theta_1 = 0$ and $\theta_2 = 1$ for $t_{n+1/2} \rightarrow t_{n+1}$. Then, the discrete version of (7.4) is

$$(7.6) \quad \begin{cases} (\frac{2}{k} + \mathcal{L})u^{n+\frac{1}{2}} = (\frac{2}{k} - \lambda + \lambda \bar{\Gamma} *)u^n \\ (\frac{2}{k} - \lambda + \lambda \bar{\Gamma} *)u^{n+1} = (\frac{2}{k} - \mathcal{L})u^{n+\frac{1}{2}}, \end{cases}$$

where $\bar{\Gamma} * u^n$ is the FFT approximation of the convolution term. As shown in [6], this scheme has the following good properties: (i) it is unconditionally stable in the von Neumann sense; (ii) for the case of deterministic parameters, the numerical solution of the scheme is locally accurate of order $O(k^2 + h^2)$; (iii) if M is the number of time steps and N is the number of steps in spatial direction, the computational burden is $O(MN \log_2 N)$.

Notice that this method is only proposed for the linear constant coefficient one dimensional case, namely for the original Merton equation. We point out

that the main difference from the scheme (7.7) that we will present in the next section, is not the FFT approximation of the convolution term. Actually, our integral approximation formula in (7.7), can be easily substituted by the FFT technique without changing the general behaviour of the scheme.

Instead, the main feature of that scheme is an original decomposition to solve the implicit part. Actually, in the second half time step of (7.6), the values $\{u_j^{n+1}\}$ are first computed in the Fourier space as

$$\langle u^{n+1} \rangle_j = \frac{\langle \left(\frac{2}{k} - \mathcal{L}\right) u^{n+\frac{1}{2}} \rangle_j}{\left(\frac{2}{k} - \lambda + \lambda \langle \Gamma \rangle_j\right)},$$

and then transformed back by the inverse FFT. However, this procedure turns out to be of difficult implementation and even the monotonicity property of the problem is far from being clear. Moreover, due to the nonlinearities and degeneracies of the equations considered, the effectiveness of these methods in the general case has still to be established.

7.1. An explicit finite difference method

In this section, we give an exhaustive description of the explicit scheme. To solve the integro-differential equation (7.4), first, we truncate the integral domain. As we have previously described in Subsection 5.2, we choose the interval $[z_m, z_M]$ such that (5.3) holds and we point out that a positive constant C exists such that

$$\int_{-\infty}^{\infty} [u(x+z, t) - u(x, t)] \Gamma_{\delta}(z) dz = \int_{z_m}^{z_M} u(x+z, t) \Gamma_{\delta}(z) dz - u(x, t) + C \delta^2 \epsilon.$$

We apply a compound rule to the integral term and a standard explicit finite-difference scheme for the differential part as done in (7.2). Then, our approximation of the equation (7.4) is given by,

$$(7.7) \quad \begin{aligned} Q(h, k, j, n, u_j^n, \mathcal{I}_h \tilde{u}, \tilde{u}) = & \frac{u_j^{n+1} - u_j^n}{k} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} \\ & - \left(\frac{q}{2k} + \frac{b}{h^2}\right) (u_{j+1}^n - 2u_j^n + u_{j-1}^n) + cu_j^n \\ & + \lambda u_j^n - \lambda \sum_{p \in P} \alpha_p u_{j+p}^n (\Gamma_{\delta})_p, \end{aligned}$$

where P is the index set of the integral approximation.

Proposition 7.1. *The scheme (7.7) is accurate to order $O\left(h^2 + \frac{qh^2}{2k}\right)$ under the CFL stability condition*

$$(7.8) \quad \frac{|a|k}{h} \leq \frac{2bk}{h^2} + q \leq 1 - (c + \lambda)k.$$

Proof. The condition (7.8) is easily checked by looking at the monotonicity of the function Q . To study the accuracy of the scheme, we use the symbol analysis [34]. Let $p(s, \xi)$ be the symbol of the integro-differential operator (7.4)

$$p(s, \xi) = s + ia\xi + b\xi^2 + c - \lambda \left(\int_{z_m}^{z_M} e^{i\xi z} \Gamma_\delta(z) dz - 1 + C\delta^2\epsilon \right).$$

Substituting $u_j^n = e^{skn} e^{ijh\xi}$ in (7.7), we get the symbol $p_{k,h}(s, \xi)$ of the difference scheme,

$$p_{k,h}(s, \xi) = \frac{e^{sk} - 1}{k} + ia \frac{\sin h\xi}{h} + 2 \left(\frac{qh^2}{2k} - b \right) \frac{\cos h\xi - 1}{h^2} + c - \lambda \left(\sum_{p \in P} \alpha_p e^{iph\xi} (\Gamma_\delta)_p - 1 \right).$$

Taking into account that our integral approximation verifies

$$\sum_{p \in P} \alpha_p e^{iph\xi} (\Gamma_\delta)_p = \int_{z_m}^{z_M} e^{i\xi z} \Gamma_\delta(z) dz + O(h^l) \text{ with } l \geq 2,$$

we have, by the Taylor expansion

$$p_{k,h}(s, \xi) = s + ia\xi + b\xi^2 + r - \lambda \left(\int_{z_m}^{z_M} e^{i\xi z} \Gamma_\delta(z) dz - 1 \right) + \frac{qh^2}{2k} \xi^2 + O(k + h^2).$$

Then, we look for a symbol $r_{k,h}(s, \xi)$ such that the difference $p_{k,h}(s, \xi) - r_{k,h}(s, \xi)p(s, \xi)$ gives the order of accuracy. We have that $r_{k,h}(s, \xi) = 1 + o(1)$ and

$$p_{k,h}(s, \xi) - p(s, \xi) = + \frac{qh^2}{2k} \xi^2 + \lambda C \delta^2 \epsilon + O(k + h^2).$$

Since $\epsilon \ll 1$ and, from the CFL condition (7.8), $k = O(h^2)$, the scheme is accurate of order $O\left(h^2 + \frac{qh^2}{2k}\right)$. \square

7.2. Numerical boundary conditions

To apply the scheme (7.7) we have to specify a numerical bounded domain. Let $\Omega \subset \mathbb{R}$ be the interval where we want to calculate the numerical solution. We set

$$\Omega_h = \left\{ x_j = jh, j \in \mathbb{Z} \mid x_j \in \Omega \right\},$$

and we define the numerical domain $\bar{\Omega}$ for the problem (7.4) in the following way. For every fixed x , set

$$\Omega_x = \left\{ z \in [z_m, z_M] \mid x + z \in \Omega \right\}$$

and

$$\bar{\Omega} = \Omega \cup \left\{ \cup_{x \in \Omega} \Omega_x^C \right\}.$$

We stress that, since the integral is a nonlocal term, its approximation will be split in two parts. For every $x \in \Omega$ fixed, we integrate on the union of the "inside" set Ω_x and the "outside" set Ω_x^C . Since accurate representation of the integral term will generally require a very wide grid, the "outside" set Ω_x^C must contain many grid points. Then, as we have defined our numerical problem on Ω , we need a limiting form for the solution u on the external set Ω^C . If $v(x, t)$ is any given analytic approximation of $u(x, t)$, the integral term will be approximated by

$$\begin{aligned} \mathcal{I}u(x) &= \lambda \left(\int_{-\infty}^{\infty} u(x+z, t) \Gamma_\delta(z) dz - u(x, t) \right) \\ &\approx \lambda \left(\int_{\Omega_x} u(x+z, t) \Gamma_\delta(z) dz \right. \\ &\quad \left. + \int_{\Omega_x^C} v(x+z, t) \Gamma_\delta(z) dz - u(x, t) \right). \end{aligned}$$

If the option price is linear in e^x , the simplest choice is to use as approximation function v the payoff function ψ .

To give a more general scheme, not depending on the initial data of the problem or on a special form of the solution, we approximate on the external set Ω^C the problem (7.4) by the diffusive one (6.2). We define

$$j_- = \inf_j \left\{ jh \in \Omega_h \right\}, j_+ = \sup_j \left\{ jh \in \Omega_h \right\},$$

$$P_{in} = P \cap \{j_-, \dots, j_+\}, P_{out} = P - P_{in}.$$

We modify the scheme (7.7) with $q = 0$ fixed, as follows

$$\begin{aligned} v_j^{n+1} &= kw_{-1}v_{j-1}^n + (1 - kw_0)v_j^n + kw_1v_{j+1}^n \\ (7.9) \quad &+ \lambda k \left[h \sum_{p \in P_{in}} \alpha_p v_{j+p}^n (\Gamma_\delta)_p + h \sum_{p \in P_{out}} \alpha_p \tilde{v}_{j+p}^n (\Gamma_\delta)_p - v_j^n \right], \end{aligned}$$

where

$$(7.10) \quad w_{-1} = \frac{b}{h^2} + \frac{a}{2h}, \quad w_0 = \frac{2b}{h^2} + c, \quad w_1 = \frac{b}{h^2} - \frac{a}{2h},$$

and where the values $\{\tilde{v}_i^n\}$ are given by the approximation of the diffusive equation (6.2) with a general diffusion coefficient D , to be fixed later,

$$(7.11) \quad \begin{aligned} \tilde{v}_i^n &= kw_{-1}v_{i-1}^{n-1} + (1 - kw_0)v_i^{n-1} + kw_1v_{i+1}^{n-1} \\ &+ \lambda k \frac{D}{h^2} (v_{i-1}^{n-1} - 2v_i^{n-1} + v_{i+1}^{n-1}). \end{aligned}$$

Let us rewrite the scheme (7.7) in the following form

$$(7.12) \quad \begin{aligned} u_j^{n+1} &= kw_{-1}u_{j-1}^n + (1 - kw_0)u_j^n + kw_1u_{j+1}^n \\ &+ \lambda k \left[h \sum_{p \in P} \alpha_p u_{j+p}^n (\Gamma_\delta)_p - u_j^n \right] - k\tau_{h,k}^\epsilon, \end{aligned}$$

where $\tau_{h,k}^\epsilon$ is the truncation error estimate in Proposition 7.1, which is $\tau_{h,k}^\epsilon = O(k + h^2) + \lambda C \delta^2 \epsilon$. We want to estimate the global difference between the two numerical solution (7.9) and (7.12). Define

$$e_j^{n+1} = v_j^{n+1} - u_j^{n+1}, \quad \text{and } E^n = \sup_j |e_j^n|.$$

We can prove the following result.

Proposition 7.2. *If $\epsilon = O(h^4)$ and for the time step there holds a standard CFL condition, $k = O(h^2)$, then $E^n = O(h^2)$, for $h \rightarrow 0$.*

Proof. Subtracting the equation (7.12) from the (7.9), we have

$$(7.13) \quad \begin{aligned} e_j^{n+1} &= kw_{-1}e_{j-1}^n + (1 - kw_0)e_j^n + kw_1e_{j+1}^n + \lambda k \left[h \sum_{p \in P_{in}} \alpha_p e_{j+p}^n (\Gamma_\delta)_p \right. \\ &+ h \sum_{p \in P_{out}} \alpha_p \left[\tilde{v}_{j+p}^n - u_{j+p}^n \right] (\Gamma_\delta)_p - e_j^n \left. \right] + k\tau_{h,k}^\epsilon. \end{aligned}$$

By (7.11) and (7.12), we obtain

$$(7.14) \quad \begin{aligned} \tilde{v}_{j+p}^n - u_{j+p}^n &= k \left(w_{-1} + \frac{\lambda D}{h^2} \right) e_{j+p-1}^{n-1} + \left[1 - k \left(w_0 + \frac{2\lambda D}{h^2} \right) \right] e_{j+p}^{n-1} \\ &+ k \left(w_1 + \frac{\lambda D}{h^2} \right) e_{j+p+1}^{n-1} + \lambda k \left[\frac{D}{h^2} (u_{j+p-1}^{n-1} - 2u_{j+p}^{n-1} + u_{j+p+1}^{n-1}) \right. \\ &\left. - \left(h \sum_{\tilde{p} \in P} \alpha_{\tilde{p}} u_{j+p+\tilde{p}}^{n-1} (\Gamma_\delta)_{\tilde{p}} - u_{j+p}^{n-1} \right) \right] + k\tau_{h,k}^\epsilon. \end{aligned}$$

We have to estimate the difference between the central second order finite difference approximation and the integral approximation.

For every $p \in P$ fixed

$$u_{j+p} - u_j = (ph)u_x + \frac{(ph)^2}{2}u_{xx} + \frac{(ph)^3}{3!}u_{xxx} + \frac{(ph)^4}{4!}u_{xxxx} + \dots$$

We call $z_\epsilon = z_M = \delta\sqrt{-2\log(\epsilon\delta\sqrt{2\pi})}$ as described in Subsection 5.2, and, for the compound rule (5.1), with $\rho = 1$, a point $\xi \in [-z_\epsilon, z_\epsilon]$ exists such that

$$h \sum_{p \in P} \alpha_p (ph)^\beta (\Gamma_\delta)_p = \int_{-z_\epsilon}^{z_\epsilon} z^\beta \Gamma_\delta(z) dz + \frac{h^2}{6} z_\epsilon \frac{d^2(z^\beta \Gamma_\delta(z))}{dz^2} \Big|_{z=\xi}.$$

Then,

$$h \sum_{p \in P} \alpha_p (u_{j+p} - u_j) (\Gamma_\delta)_p = \frac{1}{2} \frac{d^2 u(x)}{dx^2} \Big|_{x=x_j} \int_{-z_\epsilon}^{z_\epsilon} z^2 \Gamma_\delta(z) dz + \frac{h^2}{6} z_\epsilon R_j.$$

This yields

$$\begin{aligned} D \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} - h \sum_{p \in P} \alpha_p (u_{j+p} - u_j) (\Gamma_\delta)_p \\ = \frac{1}{2} \frac{d^2 u(x)}{dx^2} \Big|_{x=x_j} \left(2D - \int_{-z_\epsilon}^{z_\epsilon} z^2 \Gamma_\delta(z) dz \right) - \frac{h^2}{6} z_\epsilon R_j. \end{aligned}$$

Now, we choose the diffusion coefficient D such that

$$(7.15) \quad D = \frac{1}{2} \int_{-z_\epsilon}^{z_\epsilon} z^2 \Gamma_\delta(z) dz.$$

Under the CFL condition (7.8) and for

$$(7.16) \quad 1 - kw_0 - k \frac{2\lambda D}{h^2} \geq 0,$$

from (7.13) and (7.14), we have

$$\begin{aligned} E^{n+1} &\leq \left[1 - kc + \lambda kh \sum_{p \in P_{in}} (\Gamma_\delta)_p - \lambda k \right] E^n \\ &\quad + \left[\lambda k (1 - kc) h \sum_{p \in P_{out}} \alpha_p (\Gamma_\delta)_p \right] E^{n-1} \\ &\quad + \lambda^2 k^2 \frac{h^2}{6} z_\epsilon h \sum_{p \in P_{out}} \alpha_p |R_{j+p}| (\Gamma_\delta)_p \\ &\quad + \lambda k^2 |\tau_{h,k}^\epsilon| h \sum_{p \in P_{out}} \alpha_p (\Gamma_\delta)_p + k |\tau_{h,k}^\epsilon|. \end{aligned}$$

This is, for some coefficients A , B and C ,

$$(7.17) \quad E^{n+1} \leq AE^n + BE^{n-1} + C.$$

As a consequence, for any $\frac{n+1}{2} \leq m < n$ we have

$$E^{n+1} \leq E^{n-2m+1} \sum_{k=0}^m \binom{m}{k} (AE)^k B^{m-k} + C \sum_{k=0}^{m-1} (A+B)^k.$$

When $n - 2m + 1 = 0$, $E^0 = 0$ it yields

$$(7.18) \quad E^{n+1} \leq C \frac{1 - (A+B)^{\frac{n+1}{2}}}{1 - (A+B)}.$$

Now, we have

$$h \sum_{p \in P_{in}, P_{out}} (\Gamma_\delta)_p \leq 2z_\epsilon \max_z \Gamma_\delta = 2z_\epsilon \Gamma \text{ and } |R_j| \leq R.$$

The CFL condition (7.16) gives $k = O(h^2)$ and $\tau_{h,k}^\epsilon = O(h^2 + \epsilon)$. Then, for $N = T/k$, the global error (7.18) is estimated by

$$E^N \leq \left(1 - \frac{T}{N}c - 2\frac{T^2}{N^2}\lambda z_\epsilon \Gamma\right)^{N/2} g(h, \epsilon),$$

where

$$g(h, \epsilon) = O\left(\frac{h^4 z_\epsilon^2 + h^4 z_\epsilon + h^2 \epsilon z_\epsilon + h^2 + \epsilon}{1 + h^2 z_\epsilon}\right).$$

As h and k go to zero, we obtain

$$\lim_{N \rightarrow \infty} (A+B)^{N/2} = \left(1 - \frac{T}{N}c - 2\frac{T^2}{N^2}\lambda z_\epsilon \Gamma\right)^{N/2} = e^{\frac{TC}{2}}.$$

Then, to get the rate of convergence as $h \rightarrow 0$, we observe that the minimal value of the function $g(\cdot, \epsilon)$ is achieved for $\epsilon = O(h^4)$.

Therefore, the conclusion follows, since

$$\begin{aligned} \lim_{N \rightarrow \infty} E^N &\leq e^{\frac{TC}{2}} g(h, h^4) \leq e^{\frac{TC}{2}} \left(-h^4 \log h^4 + h^4 \sqrt{-\log h^4} \right. \\ &\quad \left. + h^6 \sqrt{-\log h^4} + h^2 + h^4\right) = O(h^2). \end{aligned}$$

□

Remark 7.3. We point out that, for the Gaussian probability density (2.2) we have

$$\bar{k} = \mathbb{E}(\eta - 1) = \exp\left(\frac{\delta^2}{2}\right) - 1 \approx \frac{\delta^2}{2} + O(\delta^4), \delta \ll 1,$$

then, solving the approximated problem (6.2) in Ω^C is just solving the Black-Scholes equation (7.1) with coefficients

$$a = \frac{\sigma^2}{2} - r + \lambda \bar{k} \approx \frac{\sigma^2}{2} - r + \lambda \frac{\delta^2}{2}, \quad b = \frac{\sigma^2}{2} + \lambda \frac{\delta^2}{2}.$$

Even if the scheme (7.9) needs for a CFL condition and its convergence in time is only first order accurate, we shall see in Subsection 8.1 that it is of simple practice application and computationally fast. It is easy to obtain a scheme which is second order in time, by applying the SSP (Strong Stability Preserving) Runge-Kutta technique, as in [24] and references therein, but we observe no real advantages for the total accuracy at least for the second order case. Then in what follows, we just use scheme (7.9).

8. Examples and Numerical tests

In this section we compute the order γ of the error in the following form

$$(8.1) \quad \gamma = \log_2 \left(\frac{e_1}{e_2} \right),$$

with

$$(8.2) \quad e_p = \frac{\|u(\frac{h}{p}, T) - u(\frac{h}{2p}, T)\|_{1,\infty}}{\|u(\frac{h}{2p}, T)\|_{1,\infty}}, \quad p = 1, 2,$$

where $u(h)$ denotes the numerical solution obtained with the space step discretization equal to h , under the discrete norm l^1 and l^∞ , respectively

$$\|u(\cdot, T)\|_1 = h \sum_i |u(x_i, T)|, \quad \|u(\cdot, T)\|_\infty = \max_i |u(x_i, T)|.$$

If not specified, in tables that follow we give the average convergence order.

8.1. European option

Let us consider the problem of pricing an European option according to the problem (7.4). As we showed in Subsection 7.1, we solve the integro-differential equation on the numerical domain $\bar{\Omega}$. We apply the second order scheme (7.9) under the CFL condition

$$(8.3) \quad h \leq \frac{2b}{a}, \quad k \leq \min \left(\frac{h^2}{2b + 2\lambda D + ch^2}, \frac{h^2}{2b + ch^2 + \lambda h^2} \right),$$

with D given by (7.15), $D = \lambda\delta^2/2$.

Let us fix the parameters as follows: $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\delta = 0.2$ and $T = 1$. In Fig. 1 we present the value of the option given by the jump-diffusion model (dotted-solid curve), the pure diffusion model ($\lambda = 0$) (solid curve), and the payoff value (dotted curve) respectively. The difference between the two models is clear: the value in the jump diffusion model is larger than the one in the pure diffusion setting in a neighborhood of the exercise price, according to the theoretical results in [31].

In Fig. 2, we show the variation of the solution according to the jump intensity λ . We compare the solution with $\lambda = 0.5$ (\diamond), $\lambda = 2$ (+) and $\lambda = 8$ (−) and we observe that the solutions increase with λ . This is what we expect from the model, because as the intensity of the jump increases the risk of the investment increases and consequently the price of the derivative needs to be higher.

In Tables T2 we show the l^∞ errors and the convergence order (8.1) for the European call option initial data. This confirms experimentally that the scheme is second order accurate.

Since we might be interested in obtaining the value u of the option for a given stock price $S^* = \exp(x^*)$, we fix $x^* = \ln(100)$ and $\sigma = 0.2$, $\delta = 0.8$, $\lambda = 0.1$. We compute the "exact" option price P by the analytical solution given in [31]. Then, in Table T3 we show the convergence order,

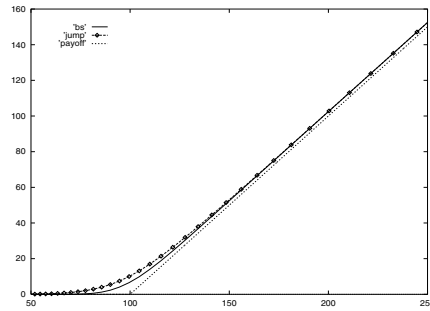


Fig. 1. Subsection 8.1, jump-diffusion model (\diamond), pure diffusion model (—) and payoff value ($\cdot\cdot\cdot$), with Simpson compound rule and $h = 0.05$

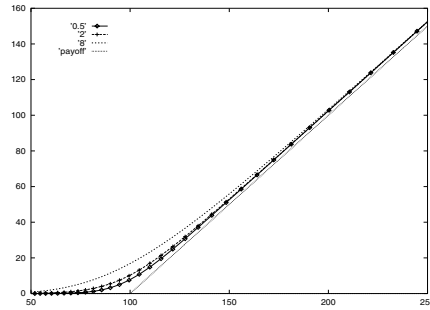


Fig. 2. Subsection 8.1, jump-diffusion model with different values of jump intensity, $\lambda = 0.5$ (\diamond), $\lambda = 2$ ($+$), $\lambda = 8$ ($-$)

Table T2. Example 8.1, l^∞ errors and convergence orders of the European call option computed using the scheme (7.9). The process parameters are $E = 100, r = 0.05, \sigma = 0.2, \lambda = 0.1, \delta = 0.8$. The number of time steps is given by the CFL condition (8.3)

N	$T = 0.1$	$T = 1$	$T = 10$
129	0.000262	0.000778	0.002118
257	0.000123	0.000122	0.000645
513	0.000044	0.000036	0.000038
1025	0.000011	0.000016	0.000130
2049	0.000000	0.000008	0.000003
Convergence-order	2.40096	1.663561	2.379236

Table T3. Example 8.1, convergence orders (8.4) with respect to the analytical solution, see [31], of the European put and call option prices of Merton model computed using the explicit scheme (7.9). The process parameters are $E = 100, r = 0.05, \sigma = 0.2, \lambda = 0.1, \delta = 0.8, x = \ln(100)$. The number of time steps is given by the CFL condition (8.3)

N	$T = 0.1$		$T = 1$		$T = 10$	
	PUT	CALL	PUT	CALL	PUT	CALL
64	1.562877	1.876357	7.809220	13.640541	13.902732	67.105924
128	2.356042	2.826212	8.167357	13.579657	14.916689	56.847952
256	2.572710	3.075204	8.268306	13.378464	14.911828	56.876483
512	2.625636	3.128717	8.319940	13.286915	15.179737	54.510145
1024	2.628921	3.122007	8.337027	13.223001	15.179249	54.513929
Analytical solution	2.633642	3.132394	8.341444	13.218501	15.179245	54.525989
Convergence-order $\tilde{\gamma}$	1.956285	2.916243	1.728216	1.637843	4.602457	2.506671

$$(8.4) \quad \tilde{\gamma} = \log_2 \left(\frac{|u(h; x^*, T) - P|}{|u(\frac{h}{2}; x^*, T) - P|} \right),$$

where $u(h; x^*, T)$ is the numerical solution of (7.9) with space step h , valued in x^* at time T . We stress out that we construct the interval Ω centered on x^* , then the values on the Table T3 are few influenced by the boundary error.

Table T4. Example 8.1, CPU times on 1.6 GHz Pentium IV PC for the scheme (7.9) when $T = 1$ and the number of time steps is given by the CFL condition (8.3). The CPU times for 64 and 128 nodes are not available

N	64	128	256	512	1024
CPU time (seconds)	N.A.	N.A.	0.02s	0.16s	4.4s

Table T4 shows CPU times on a 1,6 GHz Pentium IV PC for various number of space steps. Although the scheme (7.9) is of explicit type, it is computationally fast.

8.2. A two-dimensional example

In this section we present an operator splitting method for the two-dimensional degenerate equation (2.6). For an extensive description of operator splitting methods, we refer to the paper [26].

The main difficulty is given by the presence of a hyperbolic direction y . For simplicity, we set $r = 0$ and we write the equation in the short form

$$\partial_t u(x, y, t) + Du(x, y, t) = \lambda Ju(x, y, t),$$

where

$$\begin{aligned} Du &= -b\partial_{xx}^2 u + a_1\partial_x u + a_2\partial_y u, \\ Ju &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x + \xi, y + \eta, t) \Gamma(\xi, \eta) d\xi d\eta - u(x, y, t), \\ \Gamma(\xi, \eta) &= \frac{1}{2\pi\delta^2} \exp\left(-\frac{1}{2\delta^2}(\xi^2 + \eta^2)\right), \end{aligned}$$

and b , a_1 and a_2 are constants.

The operator splitting method can be summarized as follows: let $v^{n+1} = \mathcal{D}v^n$ be the numerical solution of

$$(8.5) \quad \partial_t v(x, y, t) + Dv(x, y, t) = 0,$$

and let $w^{n+1} = \mathcal{J}w^n$ be the numerical solution of

$$(8.6) \quad \partial_t w(x, y, t) = \lambda Jw(x, y, t).$$

Then the operator splitting is based on the following approximation

$$u^{n+1} = [\mathcal{J}\mathcal{D}]u^n.$$

To approximate the differential part (8.5), we shall apply an ADI method that combine Crank-Nicholson scheme in the two directions. To approximate the integral part (8.6), we shall apply the Euler rule for the time discretization

and the compound Simpson's product rule to the two-dimensional integral. Let us define the two following discrete operators,

$$\mathcal{D}_x = \left[\frac{k}{2} \alpha_1 \delta_x - \frac{k}{2} \beta \delta_{xx} \right], \quad \mathcal{D}_y = \left[\frac{k}{2} \alpha_2 \delta_y \right],$$

where

$$\alpha_1 = \frac{a_1}{2h_1}, \quad \alpha_2 = \frac{a_2}{h_2}, \quad \beta = \frac{b}{h_1^2} \quad \delta_{xx} u_{i,j} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$\delta_y u_{i,j} = \begin{cases} (u_{i,j} - u_{i,j-1}) & a_2 \geq 0 \\ (u_{i,j+1} - u_{i,j}) & a_2 < 0 \end{cases} \quad \delta_x u_{i,j} = (u_{i+1,j} - u_{i-1,j})$$

We observe that we are using a central finite difference scheme for the x -direction and an upwind approximation for the degenerate one. We can now write the complete scheme. To calculate the numerical solution u^{n+1} from u^n we have to solve the following two steps:

- (1) We compute the values $\tilde{u}_{i,j}^{n+1}$, for $i, j = 0, \dots, N$, by solving two tridiagonal system,

$$(8.7) \quad \begin{cases} (I + \mathcal{D}_x) \tilde{u}^{n+\frac{1}{2}} = (I - \mathcal{D}_y) u^n \\ (I + \mathcal{D}_y) \tilde{u}^{n+1} = (I - \mathcal{D}_x) \tilde{u}^{n+\frac{1}{2}} \end{cases}$$

where I denotes the identity matrix.

- (2) We obtain the solution $u_{i,j}^{n+1}$, for $i, j = 0, \dots, N$, by the expression

$$(8.8) \quad u_{i,j}^{n+1} = (1 - \lambda k) \tilde{u}_{i,j}^{n+1} + \lambda k h_1 h_2 \sum_{l,m} \alpha_l \alpha_m \tilde{u}_{i+l,j+m}^{n+1} \Gamma_{l,m}.$$

We consider the following example,

$$(8.9) \quad \begin{cases} \partial_t u(x, y, t) + Du(x, y, t) = \lambda Ju(x, y, t) & (x, y, t) \in Q \times [0, T] \\ u(x, y, 0) = u_0(x, y) & (x, y) \in Q, \end{cases}$$

where we fix the parameters $\lambda = 1$, $b = 1$, $a_1 = -a_2 = 0.5$ and the x space discretization h_1 equal to the y space discretization h_2 .

In Table T5, we show the γ order (8.1) under the norm l^1 of the scheme (8.7)-(8.8) applied to the problem (8.9) with a regular initial data $u_0(x, y) = \sin(\pi(x + y))$. We point out that we have chosen an upwind approximation to deal with the pure hyperbolic direction y , then the scheme is at most first order accurate, as well verified in Table T5.

Table T5. Convergence order γ , defined in (8.1), and errors, for the solution of the problem (8.9) with $u_0(x, y) = \sin(\pi(x + y))$, $b = 1$, $a_1 = -a_2 = 0.5$

$h_1 = h_2$	$\delta = 10^{-4}$		$\delta = 10^{-2}$	
	γ	e_p	γ	e_p
0.025		0.099739		0.089381
0.0125	1.308349	0.040273	1.485643	0.031917
0.00625	0.557111	0.027372	0.9416737	0.016169
0.003125	0.9288881	0.014378	0.447856	0.011854

8.3. The nonlinear case

As we have already seen in Section 2, the option pricing in large investor economy leads to a quasilinear differential problem. From equation (2.4), by the standard change of variable $x = \log S$, we get the following general equation

$$u_t + \mathcal{L}_{\mathcal{I}u}u = H(x, t, u, \mathcal{I}u, Du),$$

where $\mathcal{L}_{\mathcal{I}}$ is a linear degenerate elliptic integro-differential operator and H is a nonlinear integro-differential Hamilton-Jacobi operator.

The numerical approximation of Hamilton-Jacobi equations has been intensively studied, both for first and second order equations. We refer again to [16, 9] for classical results and to [32, 29, 28] for recent developments of high order accurate schemes, such as ENO, WENO, and central schemes.

Let us introduce some standard notations:

$$u^\pm = \Delta_\pm u_j = \frac{\pm(u_{j\pm 1} - u_j)}{h},$$

$$\Delta^2 u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}, \quad \hat{H}(u^+, u^-),$$

where \hat{H} is a Lipschitz continuous numerical flux, which is monotone and consistent with H [16], i.e.:

$$\hat{H}(p, p) = H(p).$$

Monotonicity here means that \hat{H} is non-increasing in its first argument and nondecreasing in the other one. Two of the most useful admissible numerical fluxes are the local Lax-Friedrichs (LLF) flux and the Godunov flux, [32].

Example 8.1. [Large institutional investor]. Let us consider the Merton model for the large investor economy. As we have seen in the Example 2.1, the interest rate r depends on the wealth ξ invested in stocks and the price function solves the quasi-linear final value problem (2.4). In the specific case of the large institutional investor, the interest rate decreases when too much wealth is invested in bonds, according to the law $r(S, t, \xi) = R(S, t)f(\xi)$ where f is a positive continuous function such that, for a given wealth $\xi_0 \geq 0$ fixed,

$f(\xi) = 1$ as $\xi \leq \xi_0$ and f is decreasing as $\xi > \xi_0$, but $f(\xi)\xi$ non decreasing. A good prototype of such type function f is given by

$$f(\xi) = \begin{cases} 1, & \xi \leq \xi_0 \\ \alpha + \beta \xi_0 \xi^{-\gamma} & \xi > \xi_0, \end{cases}$$

for all α, β and γ such that $\alpha, \beta > 0, 0 < \gamma \leq 1$ and $\alpha + \beta \xi_0^{-\gamma+1} = 1$. We select,

$$\gamma = \frac{1}{2}, \beta = \frac{1}{2\sqrt{\xi_0}} \Rightarrow \alpha = \frac{1}{2},$$

and we fix constant the interest rate $R(S, t) = R$.

We want to solve the following one dimensional quasi-linear problem,

$$(8.10) \quad \begin{cases} u_t - bu_{xx} + au_x + H(u, u_x, \int u(x+z, t) \Gamma_\delta(z) dz) = \mathcal{I}u, \\ u(x, 0) = \psi(x), \end{cases}$$

where $a = (\lambda \bar{k} + \frac{\sigma^2}{2})$, $b = \sigma^2/2$ and the non linear term H is given by

$$H(u, p, q) = Rf(u - \tilde{a}u_x - \tilde{b}(q - u))(u - \tilde{a}u_x - \tilde{b}(q - u)),$$

We stress that the H operator verify the general assumptions **F1**, **F2**, **F3**, given in Section 3, then the Cauchy problem (8.10) has a unique viscosity solution in the sense of Definition 3.3. Moreover, $H(\cdot, p, \cdot)$ is a decreasing monotone function, convex for $\xi > \xi_0$.

To discretize the equation (8.10) we approximate the nonlinear term by

$$\hat{H}_J(u_j^n, u^+, u^-, \sum_{p \in P} \alpha_p u_{j+p}^n(\Gamma_\delta)_p) = H(u_j^n, \frac{u^+ + u^-}{2}, \sum_{p \in P} \alpha_p u_{j+p}^n(\Gamma_\delta)_p),$$

This is of course a Lipschitz continuous numerical flux, monotone and consistent with $H(\cdot, p, \cdot)$. Applying the explicit scheme (7.7) for the linear part, we get the following approximation: for $j = j_-, \dots, j_+$,

$$(8.11) \quad u_j^{n+1} = u_j^n - \frac{ak}{2h} \Delta_- u_j^n + \frac{bk}{h^2} \Delta^2 u_j^n - \lambda k u_j^n + \lambda k \sum_{p \in P} \alpha_p u_{j+p}^n(\Gamma_\delta)_p - k \hat{H}_J(u_j^n, u^+, u^-, \sum_{p \in P} \alpha_p u_{j+p}^n(\Gamma_\delta)_p).$$

The scheme verifies the general convergence result (4.2) under the following CFL condition,

$$h \leq \frac{2b}{a}, \frac{2bk}{h^2} + \lambda k + k \max_u \left[\frac{dH}{du}(u, \cdot, \cdot) \right] - k \min_q \left[\frac{dH}{dq}(\cdot, \cdot, q) \right] \leq 1,$$

As it has been done for the linear problem (7.4), on the numerical boundary domain Ω^C we approximate the integral term $\mathcal{I}u$ in (8.10) by the diffusive one Du_{xx} and we solve the following equation,

$$u_t - bu_{xx} + au_x + H(u, u_x, Du_{xx}) = \lambda Du_{xx}, (x, t) \in \Omega^C \times (0, T],$$

under the condition

$$(8.12) \quad \frac{2bk}{h^2} + \frac{2\lambda Dk}{h^2} + k \max_u \left[\frac{dH}{du}(u, \cdot, \cdot) \right] - Dk \min_q \left[\frac{dH}{dq}(\cdot, \cdot, q) \right] \leq 1.$$

We fix $\tilde{a} = a$, $\tilde{b} = b$, the parameters $E = 100$, $R = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.4$ and the initial data $\psi(x) = (e^x - E)_+$ as the call option payoff function.

In Tables T7 we show the l^∞ errors and the convergence order (8.1) for $\xi_0 = 10^2$. This experimentally shows that the scheme is second order accurate.

Figure 3 shows the call option payoff function compared with the solution of (8.10) at time $T = 1$, with $\xi_0 = 10^2$ fixed.

Table T7. Example 8.1, l^∞ errors and convergence orders of the European call option computed using the scheme (8.11). The process parameters are $\xi_0 = 10^2$, $E = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.1$, $\delta = 0.4$. The number of time steps is given by the CFL condition (8.12)

N	$T = 0.5$	$T = 1$	$T = 10$
65	0.007924	0.013589	0.032857
129	0.000850	0.002836	0.007711
257	0.000610	0.000072	0.000522
513	0.000071	0.000056	0.000052
1025	0.000033	0.000068	0.000013
Convergence-order	1.977426	1.911010	2.839315

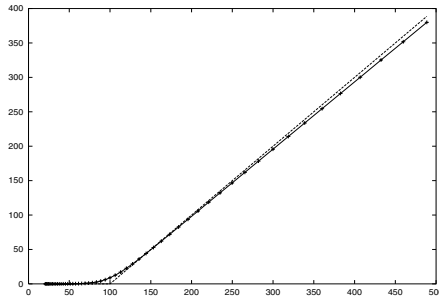


Fig. 3. Example 8.1, we show the call option payoff function compared with the solution of (8.10) (\times) at time $T = 1$, with $\xi_0 = 10^2$ fixed

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