NONLINEAR TRANSMISSION PROBLEMS FOR QUASILINEAR PARABOLIC SYSTEMS

F. R. GUARGUAGLINI† AND R. NATALINI◇

Abstract. We study degenerate quasilinear parabolic systems in two different domains, which are connected by a nonlinear transmission condition at their interface. For a large class of models, including those modeling pollution aggression on stones and chemotactic movements of bacteria, we prove global existence, uniqueness and stability of the solutions.

1. Introduction

In this paper we study degenerate quasilinear parabolic systems in two adjoining domains, with nonlinear transmission conditions at the interface. Let $N \leq 3$; let Ω_1, Ω_2 be bounded open C^2 subsets of $\mathbb{R}^N$, such that $\Omega_1 \cap \Omega_2 = \varnothing$ and let $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \neq \varnothing$ be a regular orientable manifold.

We consider the following system

$$
\left\{
\begin{array}{rll}
\partial_t(\varphi_1(c)s) & = \operatorname{div}(\varphi_1(c)\nabla s) + F_1(s,c)\ , & \\
 & & (x,t) \in \Omega_1 \times (0,T)\ ,\\
\partial_t c & = G_1(s,c)\ , & \\
 & & \\
\partial_t(\varphi_2(b)r) & = \operatorname{div}(\varphi_2(b)\nabla r) + F_2(r,b)\ , & \\
 & & (x,t) \in \Omega_2 \times (0,T),\\
\partial_t b & = G_2(r,b)\ , &
\end{array}
\right.
\tag{1.1}
$$

complemented with the boundary conditions

$$
\begin{array}{ll}
\frac{\partial s}{\partial n_1} = \psi_1(s(x,t), r(x,t), c(x,t)) & \\
 & (x,t) \in \Gamma \times (0,T)\ ,\\
\frac{\partial r}{\partial n_2} = \psi_2(r(x,t), s(x,t), b(x,t)) &
\end{array}
\tag{1.2}
$$

$$
\begin{array}{ll}
\frac{\partial s}{\partial n_1} = 0, & (x,t) \in \partial\Omega_1 \setminus \Gamma \times (0,T)\ ,\\
\frac{\partial r}{\partial n_2} = 0, & (x,t) \in \partial\Omega_2 \setminus \Gamma \times (0,T).
\end{array}
\tag{1.3}
$$

Here n_1, n_2 are the outer normal vectors respectively to $\partial\Omega_1$ and $\partial\Omega_2$, so $n_2 = -n_1$ on Γ; $\varphi_i, F_i, G_i, \psi_i$, $i = 1,2$, are given smooth functions, which verify some assumptions which will be specified later on.

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Nonlinear systems of the form (1.1) arise in models of biological and chemical phenomena like sulphation in calcium carbonate stones, chemotaxis and angiogenesis processes and in the case of a single domain have been largely studied; see for istance [10, 11, 12, 13, 5] and references therein. On the other hand, some results are available for linear parabolic equations in more domains, with linear and nonlinear conditions at interfaces, for biological models for the transfer of chemicals through semipermeable thin membranes [6, 18, 19, 4]; moreover, a semilinear system with nonlinear conditions at interfaces is studied in [21].

In the present case transmission conditions can allow to deal with models including chemical phenomena in materials with different porosity and diffusivity, and chemotaxis phenomena in regions with different substrate properties.

The aim of this paper is the proof of the global existence and uniqueness of weak solutions to transmission problem (1.1), under suitable general assumptions on F_i, G_i, φ_i and ψ_i.

In [12], in the case of one domain, we showed that assuming a priori L^∞ bounds for solutions, allows the control of the growth in time of the Sobolev norms, so proving global existence estimates. Here we follows the same approach, however, due to nonlinear conditions at the interface, we use different techniques.

The paper is organized as follows. In the next section we prove a priori estimates for the problem in one domain, following the ideas of [16], but with some nontrivial changes due to the nonlinearity of the interface conditions, which is the main concern of this paper. In Section 3 we extend these estimates to the problem in two domains and, by compactness thecniques, we obtain the local existence result for solutions related to smooth data. Section 4 is devoted to the global existence result and to a stability result which is the key to prove uniqueness of solutions and so existence in the case of more general initial data, satisfying the assumptions introduced below. Finally, in the last section, we present two specific reaction-diffusion systems, modeling respectively sulphation and chemotactic phenomena, with particular nonlinear conditions at interface, derived from the Kedem-Katchalsky equations [15] and we show that our results provide global existence and uniqueness of solutions for these models.

Assumptions

The main assumptions of this paper is the availability of a priori L^∞ bounds for the solutions; more precisely we assume that

(LIB) for all $T > 0$, it is possible to determine some positive quantities, S^T_∞, C^T_∞, R^T_∞, B^T_∞, depending on the initial data, such that

$$0 \le s(x,t) \le S^T_\infty\ ,\ 0 \le c(x,t) \le C^T_\infty \ \text{ a.e. in } \Omega_1 \times (0,T)$$

$$0 \le r(x,t) \le R^T_\infty\ ,\ 0 \le b(x,t) \le B^T_\infty \ \text{ a.e.in } \Omega_2 \times (0,T).$$

Next we write in detail the assumptions to be verified by the data and by the functions $\varphi_i, F_i, G_i, \psi_i$, $i = 1, 2$.

Let $P = 3$ if $N = 3$, $P > 2$ if $N = 2$, $P = 2$ if $N = 1$. The data c_0, b_0, s_0, r_0 are nonnegative functions such that

$$\begin{aligned} &s_0 \in W^{2,\frac{P}{2}}(\Omega_1) \cap L^\infty(\Omega_1)\ ,\quad c_0 \in W^{1,P+2}(\Omega_1) \cap L^\infty(\Omega_1)\ ,\\ &r_0 \in W^{2,\frac{P}{2}}(\Omega_2) \cap L^\infty(\Omega_2)\ ,\quad b_0 \in W^{1,P+2}(\Omega_2) \cap L^\infty(\Omega_2)\ . \end{aligned} \tag{1.4}$$

Moreover we assume that:

a) G_i, F_i, $i = 1, 2$, are continuous functions defined over $(\overline{\mathbb{R}^+})^2$ with their first and second derivatives defined over $(\overline{\mathbb{R}^+})^2$ and bounded over bounded intervals $I \subset (\overline{\mathbb{R}^+})^2$;

b) the functions $\varphi_i, \varphi_i', \varphi_i''$, $i = 1, 2$, are defined over $(\overline{\mathbb{R}^+})^2$ and bounded over bounded intervals $I \subset \overline{\mathbb{R}^+}$;

c) for all $T > 0$ there exists $\varphi_m^T > 0$ such that $\varphi_1(c), \varphi_2(b) > \varphi_m^T$ for $c \in [0, C_\infty^T]$ and $b \in [0, B_\infty^T]$.

We remark that, verifying the following conditions often turns out to be simpler than verifying asumption *(LIB)* (see Section 5):

i) for $0 \le s \le S_\infty^T$ the solutions of the second equation in (1.1) satisfy the bounds $0 \le c \le C_\infty^T$;

ii) for $0 \le r \le R_\infty^T$ and $0 \le c \le C_\infty^T$ the solutions of the first equation in (1.1) satisfy the bounds $0 \le s \le S_\infty^T$;

iii) for $0 \le r \le R_\infty^T$ the solutions of the fourth equation in (1.1) satisfy the bounds $0 \le b \le B_\infty^T$;

iv) for $0 \le s \le S_\infty^T$ and $0 \le b \le C_\infty^T$ the solutions of the third equation in (1.1) satisfy the bounds $0 \le r \le R_\infty^T$.

In the following of the paper, it turns out to be clear that the above four conditions are equivalent to condition *(LIB)*.

Finally we introduce the definition of weak solution to problem (1.1)-(1.4). Let $Q_i^T := \Omega_i \times (0, T)$, $i = 1, 2$.

Definition 1.1. *A set of functions (s, c, r, b) is a weak solution to system (1.1)–(1.4) in $\Omega \times [0, T]$ if*

a)
$$s \in C([0, T]; L^2(\Omega_1)) \cap L^2((0, T); H^1(\Omega_1)) \cap L^\infty_{loc}(Q_1^T),$$
$$r \in C([0, T]; L^2(\Omega_2)) \cap L^2((0, T); H^1(\Omega_2)) \cap L^\infty_{loc}(Q_2^T);$$

b) *for all $\gamma \in C_0^1(\overline{\Omega}_i \times [0, T))$, for (u, g, v, i) respectively equal to $(s, c, r, 1)$ and to $(r, b, s, 2)$, the following equality holds*
$$\int_{Q_i^T} (\varphi_i(g) u \gamma_t - \varphi_i(g) \nabla u \nabla \gamma + F_i(u, g)\gamma) \ dx\, dt$$
$$+ \int_{\Omega_i} \varphi(g_0(x)) u_0(x) \gamma(x, 0) \ dx + \int_0^T \int_\Gamma \varphi_i(g) \psi_i(u, v) \gamma d\sigma dt = 0 \ ;$$

c)
$$c \in C([0, T]; L^2(\Omega_1)) \cap L^2((0, T); H^1(\Omega_1)) \cap L^\infty_{loc}(Q_1^T),$$
$$b \in C([0, T]; L^2(\Omega_2)) \cap L^2((0, T); H^1(\Omega_2)) \cap L^\infty_{loc}(Q_2^T) \ ;$$

d) *for (g, u, i) respectively equal to $(c, s, 1)$ and to $(b, r, 2)$, the following equality holds*
$$g(x, t) = g_0(x) + \int_0^t G_i(u(x, \tau), g(x, \tau)) d\tau \ ,$$
for a.e. $x \in \Omega_i$ and all $t \in [0, T]$.

2. A PRIORI ESTIMATES IN ONE DOMAIN

The proofs of the main results of this paper involve the study of the problem described below, in a single domain, which is reviewed and sligthly extended in the present section.

Let Ω be a bounded open C^2 subset of $\mathbb{R}^N$ and let $Q^T := \Omega \times (0, T)$.

In this section we are interested in the proof of some a priori estimates for the solutions u, g of the following problems:

A) the ordinary differential problem

$$\begin{cases} \partial_t g = G(f,g) \qquad (x,t) \in Q^T, \\ \\ g(x,0) = g_0(x) \qquad x \in \Omega \ ; \end{cases} \tag{2.1}$$

B) the semilinear parabolic problem

$$\begin{cases} \varphi(g)\partial_t u = \nabla(\varphi(g)\nabla u) + F(u,g) - u\varphi'(g)G(u,g), \quad (x,t) \in Q^T, \\ \\ u(x,0) = u_0(x) \geq 0 \ , \ x \in \Omega \ , \\ \\ \frac{\partial u}{\partial n} = \psi(u(x,t), v(x,t), g(x,t)) \ , \qquad (x,t) \in \Gamma_1 \times (0,T) \ , \\ \\ \frac{\partial u}{\partial n} = 0 \ , \qquad (x,t) \in \Gamma_2 \times (0,T) \ , \end{cases} \tag{2.2}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

Our first step is to prove some a priori estimates for the solution of (2.1).

We introduce the following set of functions

$$\begin{aligned} W^q(\Omega,T) = C([0,T]; W^{2,q}(\Omega)) \cap C^1([0,T]; L^q(\Omega)) \\ \cap L^1((0,T); W^{2,q+1}(\Omega)) \cap W^{1,1}((0,T); L^{q+1}(\Omega)) \ ; \end{aligned}$$

we remark that $W^q(\Omega,T) \cap L^\infty(Q^T) \subset C([0,T]; W^{1,2q}(\Omega)) \cap L^1((0,T); W^{1,2(q+1)}(\Omega))$. We shall work in such space since, for q large enough, it ensures the smoothness of the coefficients in the equations for s and r, which is necessary to perform the computations which follow.

Finally, let $f, v \in W^P(\Omega,T) \cap L^\infty(Q^T)$, $g_0 \in W^{2,P+1}(\Omega)$ and $u_0 \in W^{2,P}(\Omega)$ be some fixed nonnegative functions and let F, G, ψ, φ be functions verifying the assumptions a)-c) in the Introduction; moreover we assum condition *(LIB)* .

The proof of the following proposition can be found in [11].

Proposition 2.1. *Let $T > 0$ and let g be the solution of (2.1) in $\Omega \times [0,T]$. Then for all $t \in [0,T]$, for all p such that $1 < p \leq P$, for $j, i = 1, ..., N$, we have*

$$\|g_{x_j}\|^p_{L^p(\Omega)} \leq \left(\|g_{0x_j}\|^p_{L^p(\Omega)} + \int_0^t \int_\Omega |G_f||f_{x_j}|^p \right) e^{A_{1p}t} \ , \tag{2.3}$$

$$\begin{aligned} \|g_{x_ix_j}\|^p_{L^p(\Omega)} \leq \Bigg(\|g_{0x_ix_j}\|^p_{L^p(\Omega)} + \\ \int_0^t \int_\Omega B_{2p} \left(|f_{x_i}|^{2p} + |f_{x_j}|^{2p} + |g_{x_i}|^{2p} + |g_{x_j}|^{2p} + |f_{x_ix_j}|^p \right) dxdt \Bigg) e^{A_{2p}t} \ , \end{aligned} \tag{2.4}$$

where

$$A_{1p} = p\|G_{\tilde{c}}\|_\infty + (p-1)\|G_f\|_\infty \ ,$$

$$A_{2p} = p\|G_{\tilde{c}}\|_\infty + (p-1)\left(\|G_f\|_\infty + 2(\|G_{ff}\|_\infty + \|G_{gg}\|_\infty + \|G_{fg}\|_\infty + \|G_{gf}\|_\infty)\right) \ ,$$

$$B_{2p} = \|G_f\|_\infty + \|G_{ff}\|_\infty + \|G_{gg}\|_\infty + \|G_{fg}\|_\infty + \|G_{gf}\|_\infty \ .$$

Lemma 2.1. *Let $T > 0$ and let (u,g) be the solution of (2.1)-(2.2) in Q^T. Then*

$$\begin{aligned} \int_\Omega \varphi(g(T))u^2(T) \ dx + 2\int_{Q^T} \varphi(g)|\nabla u|^2 \ dx\,dt = \int_\Omega \varphi(g_0)u_0^2 \ dx \\ +2\int_0^T \int_{\Gamma_1} \varphi(g)u\psi(u,v,g) \ dx\,dt + \int_{Q^T} u((\varphi'(g)G(f,g) - 2G(u,g))u + 2F(u,g)) \ dx\,dt \ . \end{aligned} \tag{2.5}$$

Proof. The claim follows immediately by multiplying by u the equation in (2.2). $\square$

The previous proposition and the results in [16] imply that problem (2.2) has a solution $u \in C([0,T];L^2(\Omega)) \cap L^2((0,T);H^1(\Omega))$. Moreover, if the data are smooth enough and satisfy standard compatibility conditions on the set $\{(x,t) : x \in \partial\Omega\ ,\ t=0\}$, u is a classical solution belonging to the space $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q^T})$.

Now, we are going to prove that, if $f, v \in W^P(\Omega,T)$ then $u \in W^P(\Omega,T)$. All the computations in the proofs of the following results are made for classical solutions and then, by density arguments on the data, extended to solutions in $W^P(\Omega,T)$. Notice that in the following estimates we are going to stress the dependence of all the constants on the data of the problem. The proofs follow the ideas in [16] (cap. 5, par.7); anyway some changes are needed, due to the dependence of function ψ on v, to avoid the assumption of boundedness in $L^\infty-$norm of the quantities $\psi_x, \psi_t, \psi_{ux}, \psi_{ut}$ and smoothness of solutions.

First we obtain the estimate for ∇u in $L^2(Q^T)$. For all sufficiently smooth functions $\eta(x,t)$ we have

$$\int_\Omega (u_t\eta + \nabla u \cdot \nabla\eta + a\eta)\, dx + \int_{\Gamma_1} \psi(u,v,g)\eta dx = 0\ ,$$

where

$$a(x,t,u,\nabla u) := -\frac{1}{\varphi(g)}\left(\varphi'(g)\nabla g \cdot \nabla u + H\right)\ ,$$
$$H := F(u,g) - \varphi'(g)G(f,g)u\ .$$

Let Γ' some small part of the boundary Γ_1; without loss of generality one can assume that Γ' lies in the plane $x_n = 0$ and Ω is in the halfspace $x_n \le 0$. For functions η which vanish on $\partial\Omega \setminus \Gamma'$, the surface integral in the previous equality can be rewritten as a volume integral (see [16] cap. 5, section 7) to obtain

$$\int_\Omega \left[u_t\eta + \nabla u\cdot\nabla\eta + \left(a + \psi_u u_{x_n} + \psi_v v_{x_n} + \psi_g g_{x_n}\right)\eta + \psi\eta_{x_n}\right] dx = 0\ . \tag{2.6}$$

We assume that $v \le V^T_\infty$, $f \le U^T_\infty$; let G^T_∞ and U^T_∞ be the upper bounds for g and u derived by the assumption *(LIB)*.

Lemma 2.2. *Let $T > 0$ and let (u,g) be the solution of (2.1)-(2.2) in Q^T. There exists a constant $K = K(U^T_\infty, \|\varphi\|_{C^1}, \|\psi\|_{C^1}, \|G\|_\infty)$such that, for all $t \in [0,T]$*

$$\int_0^T\int_\Omega \left(u_t^2 + |\nabla u|^4 + |\Delta u|^2\right) dxdt + \int_\Omega |\nabla u(t)|^2 dx \le K\left(1 + \int_\Omega |\nabla u_0|^2 dx \right.$$
$$\left. + \int_0^t\int_\Omega |\nabla u|^2 dxdt + \int_0^t\int_\Omega \left(|\nabla g|^4 + |\nabla g|^2 + H^2 + |\nabla v|^2 + v_t^2 + 1\right) dxdt\right)\ .$$

Proof. Let Γ' be a small part of $\partial\Omega$ and let z be a function vanishing on $\partial\Omega \setminus \Gamma'$. We set $\eta = u_t z^2$ in (2.6), and we obtain

$$\int_0^t\int_\Omega u_t^2 z^2 dxdt + \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + 2\psi u_{x_n}\right) z^2 dx\Bigg|_0^t$$
$$= \int_0^t\int_\Omega |\nabla u|^2 z z_t dxdt + \int_0^t\int_\Omega \Big[-2u_t z\nabla u\cdot\nabla z - (a + \psi_v v_{x_n} + \psi_g g_{x_n})u_t z^2$$
$$+ u_{x_n}\left(\psi_v v_t z^2 + \psi_g g_t z^2 + 2\psi z z_t\right) - 2\psi u_t z z_{x_n}\Big] dxdt\ .$$

By using Young inequality and the assumptions on ψ we deduce the following estimate

$$\int_0^t\int_\Omega u_t^2 z^2 dxdt + \int_\Omega |\nabla u(t)|^2 z^2 dx \le C\int_\Omega \left[(\nabla u_0^2 + 1)z^2(x,0) + z^2(x,t)\right] dx$$
$$+ C_z\left(1 + \frac{1}{\epsilon}\right)\int_0^t\int_\Omega |\nabla u|^2 dxdt + \epsilon C\int_0^t\int_\Omega u_t^2 z^2 dxdt + \epsilon C\int_0^t\int_\Omega |\nabla u|^4 z^2 dxdt$$

$$+C\int_0^t\int_\Omega\left(\frac{1}{\epsilon^3}|\nabla g|^4+\frac{1}{\epsilon}|\nabla g|^2+H^2+1\right)z^2dxdt$$

$$+C\int_0^t\int_\Omega\left(\frac{1}{\epsilon}v_{x_n}^2+v_t^2\right)z^2dxdt+C\int_0^t\int_\Omega\left(z_t^2+|\nabla z|^2\right)dxdt\ ,$$

where ϵ is a small positive number and C is a constant depending on $\|\psi\|_{C^1}$, $\|G\|_\infty$ and C_z is a constant depending also on $\|z\|_{C^1}$.

Standard arguments give

$$\begin{aligned}\int_0^t\int_\Omega u_t^2dxdt+\int_\Omega|\nabla u(t)|^2dx\le C_1\int_\Omega\nabla u_0^2dx+C_{1z}\\ +C_z\left(1+\frac{1}{\epsilon}\right)\int_0^t\int_\Omega|\nabla u|^2dxdt+\epsilon C_1\int_0^t\int_\Omega u_t^2dxdt+\epsilon C_1\int_0^t\int_\Omega|\nabla u|^4dxdt\\ +C_1\int_0^t\int_\Omega\left(\frac{1}{\epsilon^3}|\nabla g|^4+H^2+\frac{1}{\epsilon}\left(|\nabla v|^2+|\nabla g|^2\right)+v_t^2+1\right)dxdt\ ,\end{aligned}\tag{2.7}$$

where C_{1z} depens on t and Ω.

Now we use the following bound for $\int_\Omega|\nabla u|^4dxdt$, coming from Gagliardo-Nirenberg inequalities,

$$\int_\Omega|\nabla u|^4dx\le C_\psi\|u\|_{L^\infty(\Omega)}^2\left(\|\Delta u\|_{L^2(\Omega)}^2+\|u\|_{W^{1,2}(\Omega)}^2\right)\ ,\tag{2.8}$$

where C_ψ depends on $\|\psi\|_{C^1}$.

By using the equation for u and the above inequality we have

$$\int_\Omega|\Delta u|^2\ dx\le C_2\left(\int_\Omega\left(|u_t|^2+|\nabla g|^4+H^2\right)dx+C_\psi(U_\infty^T)^2\|u\|_{W^{1,2}(\Omega)}^2\right)\ ,\tag{2.9}$$

where C_2 depends on φ. Now, using again (2.8), we can obtain the desidered estimate

$$\int_\Omega|\nabla u|^4dx\le C_3\int_\Omega\left(|u_t|^2+|\nabla c|^4+u^2+|\nabla u|^2+H^2\right)dx\ ,\tag{2.10}$$

where C_3 depends on C_2,C_ψ,U_∞^T.

Now the estimates (2.9) and (2.10), together with (2.7), prove the claim. □

Lemma 2.3. *Let $T>0$ and let (u,g) be the solution of (2.1)-(2.2) in Q^T. For $p\ge 2$ and for suitable small ϵ , for all $t\in[0,T]$,*

$$\begin{aligned}\int_\Omega\left(|u_t(t)|^p+|\nabla u(t)|^{2p}\right)dx+\int_0^t\int_\Omega|u_t|^{p-2}|\nabla u_t|^2dxdt+\int_0^t\int_\Omega|\nabla u|^{2(p+1)}dxdt\\ +\int_0^t\int_\Omega|\Delta u|^{p+1}\ dxdt+\int_0^t\int_\Omega|u_t|^{p+1}dxdt\\ \le C_6\Bigg(|\Omega|t+\int_\Omega\left(|\Delta u_0|^p+|\nabla g_0|^{2p}+|\nabla u_0|^{2p}+1\right)dx\\ +\int_0^t\int_\Omega\left(|\nabla g|^{2(p+1)}+|\nabla g|^{p+1}+|\nabla v|^{p+1}+|u_t|^p+|\nabla u|^{p+1}+|\nabla f|^{p+1}\right)dxdt\\ +\int_0^t\int_\Omega\left(|\nabla v_t|^{\frac{p+1}{2}}+\frac{1}{\epsilon}|v_t|^{\frac{2(p+1)}{3}}+\epsilon|\nabla v|^{2(p+1)}\right)dxdt\Bigg)\ .\end{aligned}$$

where C_6 depends on the quantity K in the previous lemma, on U_∞^T, G_∞^T and the range of the values assumed by the functions F,G,φ,ψ and their derivatives.

Proof. Let Γ' be a small part of $\partial\Omega$ and z be a function vanishing on $\partial\Omega \setminus \Gamma'$. We derive with respect to t the equation for u, then we multiply it by $|u_t|^{p-2}u_t z^2$ and we make an integration by parts; the same procedure used in the proof of the previous lemma allows to rewrite the surface integral in a volume integral and to obtain

(2.11)

$$\frac{1}{p}\int_\Omega |u_t|^p z^2 dx\bigg|_0^t$$

$$+\sum_{i=1}^N \int_0^t \int_\Omega \left(u_{x_i t} + (\psi_u u_t + \psi_v v_t + \psi_g g_t)\delta_n^i\right)\left((p-1)|u_t|^{p-2}u_{tx_i}z^2 + |u_t|^{p-2}u_t 2zz_{x_i}\right)dxdt$$

$$+\int_0^t \int_\Omega \left(a_t + a_u u_t + \sum_{i=1}^N a_{u_{x_i}} u_{tx_i} + \psi_{uu}u_t u_{x_n}\right)|u_t|^{p-2}u_t z^2 dxdt$$

$$+\int_0^t \int_\Omega \left(\psi_{uv}(v_{x_n}u_t + v_t u_{x_n}) + \psi_{ug}(g_{x_n}u_t + g_t u_{x_n})\right)|u_t|^{p-2}u_t z^2 dxdt$$

$$+\int_0^t \int_\Omega \left(\psi_{vv}v_t v_{x_n} + \psi_{vg}v_t g_{x_n} + \psi_v v_{tx_n}\right)|u_t|^{p-2}u_t z^2 dxdt$$

$$+\int_0^t \int_\Omega \left(\psi_{gg}g_t g_{x_n} + \psi_{gv}g_t v_{x_n} + \psi_g g_{tx_n}\right)|u_t|^{p-2}u_t z^2 dxdt = 0\ .$$

Taking into account that $|a_t| \le C(|\nabla u|(|\nabla f| + |\nabla g|) + 1)$, where C depends on U^T_∞, G^T_∞ and on the range of the values assumed by the functions F, G, φ, ψ, and using repeatedly the Young inequality we have the following inequality, for $p \ge 2$ and $\epsilon < 1$,

(2.12)

$$\int_\Omega |u_t|^p z^2 dx|_0^t + \int_0^t \int_\Omega |u_t|^{p-2}|\nabla u_t|^2 z^2 dxdt$$

$$\le C_1\left(\int_0^t \int_\Omega |u_t|^p\left(|\nabla z|^2 + z^2\left(1 + |\nabla g|^2 + |\nabla u| + |\nabla v| + \epsilon|\nabla v|^2 + \epsilon|\nabla u|^2\right)\right)dxdt\right.$$

$$+\int_0^t \int_\Omega |u_t|^{p-1}z^2\left(1 + |\nabla g|^2 + |\nabla u|^2 + |\nabla f|^2 + |\nabla v_t|\right)dxdt$$

$$\left.+\left(1+\frac{1}{\epsilon}\right)\int_0^t \int_\Omega |u_t|^{p-2}(|v_t|^2+1)z^2 dxdt\right)\ ,$$

where C_1 depends on U^T_∞, G^T_∞ and on the range of the values assumed by the functions F, G, φ, ψ; by using again the Young inequality we obtain

$$\int_\Omega |u_t(t)|^p z(t)^2 dx\bigg|_0^t + \int_0^t \int_\Omega |u_t|^{p-2}|\nabla u_t|^2 z^2 dxdt \le I_1 \tag{2.13}$$

where

$$
\begin{aligned}
I_1 := C_2 \Bigg(& (\sigma+\epsilon) \int_0^t \int_\Omega |u_t|^{p+1} z^2 dxdt \\
& + \int_0^t \int_\Omega \left(|u_t|^{p-1} + \frac{|u_t|^{p-2}}{\epsilon} \right) z^2 dxdt + \int_0^t \int_\Omega |u_t|^p \left(|\nabla z|^2 + z^2 \right) dxdt \\
& + C_\sigma \int_0^t \int_\Omega \left(|\nabla g|^{p+1} + |\nabla u|^{p+1} + |\nabla f|^{p+1} + |\nabla v|^{p+1} + |\nabla g|^{2(p+1)} \right) z^2 dxdt \\
& + \left(1 + \frac{1}{\epsilon}\right) C_\sigma \int_0^t \int_\Omega |v_t|^{\frac{2(p+1)}{3}} z^2 dxdt + C_\sigma \int_0^t \int_\Omega |\nabla v_t|^{\frac{p+1}{2}} z^2 dxdt \\
& \epsilon \int_0^t \int_\Omega (|\nabla u|^{2(p+1)} + |\nabla v|^{2(p+1)}) z^2 dxdt \Bigg) \; ;
\end{aligned}
$$

here C_σ increases to infinity when σ goes to zero and C_2 depends on U_∞^T, G_∞^T and on the range of the values assumed by the functions F, G, φ, ψ.

Now, setting $\eta = |u_t|^{p-1} u_t z^2$ in (2.6) we have the further estimate, for $\nu < 1$,

$$
\begin{aligned}
\int_0^t \int_\Omega |u_t|^{p+1} z^2 dxdt \le C_3 \Bigg(& C_\nu \int_0^t \int_\Omega |u_t|^{p-2} |\nabla u_t|^2 z^2 dxdt \\
+ \int_0^t \int_\Omega |u_t|^p \left(\nu |\nabla u|^2 + C_\nu |\nabla g|^2 + |\nabla v| + 1 \right) z^2 dxdt & + C_\nu \int_0^t \int_\Omega |u_t|^p |\nabla z|^2 dxdt \Bigg) \; ,
\end{aligned}
$$

where C_ν increases to infinity when ν goes to zero and C_3 depends on U_∞^T, G_∞^T and on the range of the values assumed by the functions F, G, φ, ψ and their derivatives; it follows that

$$
(1-\nu) \int_0^t \int_\Omega |u_t|^{p+1} z^2 dxdt \le C_4 C_\nu \int_0^t \int_\Omega |u_t|^{p-2} |\nabla u_t|^2 z^2 dxdt + I_2
$$

where

$$
\begin{aligned}
I_2 = C_4 \Bigg(& \int_0^t \int_\Omega \left(C_\nu (|\nabla g|^{2(p+1)} + |\nabla v|^{p+1}) + \nu |\nabla u|^{2(p+1)} \right) z^2 dxdt \\
& + \int_0^t \int_\Omega |u_t|^p \left(z^2 + C_\nu |\nabla z|^2 \right) dxdt \Bigg)
\end{aligned}
$$

and C_4 depends on U_∞^T, G_∞^T and on the range of the values assumed by the functions F, G, φ, ψ and their derivatives. Thanks to (2.13) we have

$$
(1-\nu) \int_0^t \int_\Omega |u_t|^{p+1} z^2 dxdt \le C_4 C_\nu I_1 + I_2 \tag{2.14}
$$

Putting together the estimates (2.13) and (2.14) we obtain, for suitable small parameters ϵ, σ, ν

$$
\begin{aligned}
\int_\Omega |u_t(t)|^p dx + & \int_0^t \int_\Omega |u_t|^{p-2} |\nabla u_t|^2 dxdt + (1 - \nu - C_2(1 + C_4 C_\nu)(\sigma+\epsilon)) \int_0^t \int_\Omega |u_t|^{p+1} dxdt \\
\le & \int_\Omega |u_{0t}|^p dx + C_4 \left(\int_0^t \int_\Omega \left(C_\nu \left(|\nabla g|^{2(p+1)} + |\nabla v|^{p+1} \right) + \nu |\nabla u|^{2(p+1)} \right) dxdt \right. \\
& + \overline{C}_\nu \int_0^t \int_\Omega \left(|u_t|^p + |u_t|^{p-1} + \frac{|u_t|^{p-2}}{\epsilon} \right) dxdt \\
+ \overline{C}_\nu \overline{C}_\sigma & \int_0^t \int_\Omega \left(|\nabla g|^{p+1} + |\nabla u|^{p+1} + |\nabla f|^{p+1} + |\nabla v|^{p+1} + |\nabla g|^{2(p+1)} + |\nabla v_t|^{\frac{p+1}{2}} \right) dxdt
\end{aligned}
$$

$$+\overline{C}_\nu \left(\frac{\overline{C}_\sigma}{\epsilon} \int_0^t \int_\Omega |v_t|^{\frac{2(p+1)}{3}} dxdt + \epsilon \int_0^t \int_\Omega (|\nabla u|^{2(p+1)} + |\nabla v|^{2(p+1)}) dxdt \right) \right) ,$$

where $\overline{C}_\nu, \overline{C}_\sigma$ increase to infinity when ν, σ go to zero.

To treat the term $|\nabla u|^{2(p+1)}$ we use the same technique used at the end of the proof of Lemma 2.2 and similarly to (2.9) and (2.10) we obtain, for $q \geq 1$

$$\begin{aligned} &\int_\Omega |\Delta u|^q \, dx \\ &\leq C_6 \left(\int_\Omega \left(|u_t|^q + |\nabla g|^{2q} + H^q \right) dxdt + C_\psi \|u\|^q_{L^\infty(\Omega)} \|u\|^q_{W^{1,q}(\Omega)} dt \right) \end{aligned} \tag{2.15}$$

and

$$\int_\Omega |\nabla u|^{2q} dx \leq C_7 \int_\Omega \left(|u_t|^q + |\nabla g|^{2q} + \|u\|^q_{W^{1,q}(\Omega)} + H^q \right) dxdt \ , \tag{2.16}$$

where the constants C_6 and C_7 depend on U^T_∞, G^T_∞ and on the range of the values assumed by the functions φ, ψ and their derivatives.

Taking into account the above estimates, for a suitable choice of small ν, σ and ϵ, we obtain the claim.

□

Starting from the result in Lemma 2.1, by using Lemmas 2.2, 2.3, interpolation results in L^p spaces and taking into account the results in Proposition 2.1, we can write, for $2 \leq p \leq 3$,

(2.17)

$$\begin{aligned} &\sup_{[0,T]} \int_\Omega \left(|u_t|^p + |\nabla u|^{2p} \right) dx + \int_0^T \int_\Omega |u_t|^{p-2} |\nabla u_t|^2 dxdt + \int_0^T \int_\Omega |\nabla u|^{2(p+1)} dxdt \\ &+ \int_0^T \int_\Omega |\Delta u|^{p+1} \, dxdt + \int_0^T \int_\Omega |u_t|^{p+1} dxdt \\ &\leq C_8 \left(\int_\Omega \left(|\Delta u_0|^p + |\nabla g_0|^{2p} + T|\nabla g_0|^{p+1} + T|\nabla g_0|^{2(p+1)} + |\nabla u_0|^{2p} + 1 \right) dx \right. \\ &+1 + \int_0^T \int_\Omega \left(T|\nabla f|^{2(p+1)} + |\nabla v|^{p+1} + |u_t|^p + (1+T)|\nabla f|^{p+1} \right) dxdt \\ &\left. + \int_0^T \int_\Omega \left(|\nabla v_t|^{\frac{p+1}{2}} + \frac{1}{\epsilon} |v_t|^{\frac{2(p+1)}{3}} + |\nabla v|^{2(p+1)} \right) dxdt \right) , \end{aligned}$$

where C_8 depends on the L^∞ norm of the data, U^T_∞, G^T_∞, the range of the values assumed by F, G, ψ, φ and their derivatives, T and Ω, increases with T and it is strictly positive for $T = 0$.

3. Local existence in two domains

Let $\Omega := \Omega_1 \cup \Omega_2$ and $T > 0$. Let Φ_i, $i = 1, 2$, be the maps which associate to the pair $(v_1, v_2) \in (W^P(\Omega, T))^2$ the pairs $(u_i, g_i)) \in (W^P(\Omega_i, T))^2$, $i = 1, 2$, solutions to the problems

(3.1)
$$\begin{cases} \varphi_i(g_i)\partial_t u_i = \mathrm{div}(\varphi_i(g_i)\nabla u_i) + F_i(u_i,g_i) - u_i\varphi_i'(g_i)G_i(u_i,g_i), \quad (x,t)\in\Omega_i\times(0,T), \\ \\ \partial_t g_i = G_i(v_i,g_i)\ , \qquad (x,t)\in\Omega_i\times(0,T), \end{cases}$$

complemented with the initial conditions

$$\begin{aligned} &g_1(x,0)=c_0(x)\geq 0\ , \quad u_1(x,0)=s_0(x)\geq 0, \quad x\in\Omega_1\ , \\ &g_2(x,0)=b_0(x)\geq 0\ , \quad u_2(x,0)=r_0(x)\geq 0, \quad x\in\Omega_2 \end{aligned} \tag{3.2}$$

and the boundary conditions, for $i=1,2$,

$$\begin{aligned} &\tfrac{\partial u_i}{\partial n_i} = \psi_i(u_i(x,t),v_i(x,t),g_i(x,t))\ , \qquad (x,t)\in\Gamma\times(0,T) \\ &\tfrac{\partial u_i}{\partial n_i} = 0\ , \qquad (x,t)\in\partial\Omega_i\setminus\Gamma\times(0,T)\ , \end{aligned} \tag{3.3}$$

where the initial data s_0, r_0 belong respectively to the spaces $W^{2,P}(\Omega_1)$ and $W^{2,P}(\Omega_2)$ and c_0, b_0 to the spaces $W^{2,P+1}(\Omega_1)$ and $W^{2,P+1}(\Omega_2)$; in order to build sequences by means of the maps Φ_i, we need to extend the functions u_i, g_i to functions belonging to $W^P(\Omega,T)$, still denoted u_i, g_i.

Let us define the quantities

$$\|u\|_m^{\Omega,T} = \sup_{[0,T]}\int_\Omega |u_t|^m dx + \sup_{[0,T]}\int_\Omega |\nabla u|^{2m} dx + \int_0^T\int_\Omega (|u_t|^{m+1} + |\nabla u|^{2(m+1)})dxdt$$

$$|||u|||_m^{\Omega,T} = \|u\|_m^{\Omega,T} + \int_0^T\int_\Omega |\nabla u_t|^2 dxdt\ .$$

The inequality (2.17), for small T , gives

(3.4)
$$|||u|||_2 \leq K^T\left(I_0(2)+1+\left(T+\epsilon+\frac{T}{\epsilon}\right)(\|v\|_2+\|f\|_2)+\epsilon\int_0^T\int_\Omega|\nabla v_t|^2dxdt+\frac{|\Omega|T}{\epsilon}\right)$$

$$\leq K^T\left(I_0(2)+1+\left(T+\epsilon+\frac{T}{\epsilon}\right)(|||v|||_2+\|f\|_2)+\frac{|\Omega|T}{\epsilon}\right)\ ,$$

where

$$I_0(p) := \int_\Omega\Big(|\Delta u_0|^p + |\nabla g_0|^{2p} + |\nabla u_0|^{2p} + T|\nabla g_0|^{p+1} + T|\nabla g_0|^{2(p+1)} + 1\Big)dx\ ;$$

moreover

(3.5)
$$\|u\|_3 \leq K^T\left(I_0(3)+1+\left(T+\epsilon+\frac{T}{\epsilon}\right)(\|v\|_3+\|f\|_3)\ +\int_0^T\int_\Omega|\nabla v_t|^2dxdt\right)\ .$$

We omitted the superscripts Ω, T for simplicity of notations. Here K^T depends on the L^∞ norm of the data, U_∞^T, G_∞^T (see Section 2), the range of the values assumed by F, G, ψ, φ and their derivatives, T and Ω, increases with T and it is strictly positive for $T=0$. We remark that the choice of small T in the above inequalities depends on the the quantity C_8 at the end of the previous section, i.e. on the L^∞ norm of the data, U_∞^T, G_∞^T, the range of the values assumed by F, G, ψ, φ and their derivatives, T and Ω.

Starting from the pair $(s^1,r^1)\in(W^P(\Omega,T))^2$, with $0\leq s^1\leq S_\infty^T$ and $0\leq r^1\leq R_\infty^T$, we define the sequences $\{s_n\}$, $\{c_n\}$, $\{r_n\}$, $\{b_n\}\subset W^P(\Omega,T)$ by recurrency

$$(s^{n+1},c^{n+1}) = \Phi_1(s^n,r^n)$$

$$(r^{n+1}, b^{n+1}) = \Phi_2(r^n, s^{n+1}) \ .$$

We are going to prove uniform estimates for such sequences, based on the a priori estimates (3.4) and(3.5).

Thanks to the assumptions, the uniform boundedness in L^∞ norm is ensured, i.e., for $t < T$,

$$0 \le s^n \le S^T_\infty \ , \quad 0 \le r^n \le R^T_\infty \ , \quad 0 \le c^n \le C^T_\infty \ , \quad 0 \le b^n \le B^T_\infty \ ;$$

next, we remark that the estimate in Lemma 2.1 depends on f and g only through the quantities $\|f\|_{L^\infty(Q^T)}$, $\|g\|_{L^\infty(Q^T)}$; hence we can write, for all $n \in \mathbb{N}$,

$$\|\nabla s^n\|_{L^2(Q^T)}, \|\nabla r^n\|_{L^2(Q^T)} \le D^T \tag{3.6}$$

where D^T is a positive constant determined by the L^∞ norm of the data, S^T_∞, C^T_∞, R^T_∞, B^T_∞, Ω and T, non decreasing with T, independent on n.

Then, after setting $\epsilon = \sqrt{T}$ and

$$\begin{aligned} J_0(p) := \int_\Omega \Big(& |\Delta s_0|^p + |\Delta r_0|^p + |\nabla c_0|^{2p} + |\nabla s_0|^{2p} + T|\nabla c_0|^{p+1} \\ & + T|\nabla c_0|^{2(p+1)} + |\nabla b_0|^{2p} + |\nabla r_0|^{2p} + T|\nabla b_0|^{p+1} + T|\nabla b_0|^{2(p+1)} + 1 \Big)\, dx \ , \end{aligned}$$

inequalities (3.4) and (3.5) give

$$\|s^{n+1}\|_3 \le K^T_1 \left(J_0(3) + 1 + \left(T + \sqrt{T}\right) (\|r^n\|_3 + \|s^n\|_3) + \int_0^T \int_\Omega |\nabla r^n_t|^2 dxdt \right) \ , \tag{3.7}$$

$$\int_0^T \int_\Omega |\nabla s^{n+1}_t|^2 dxdt \le K^T_1 \left(J_0(2) + \left(T + \sqrt{T}\right) (|||r^n|||_2 + \|s^n\|_2) + 1 \right) \tag{3.8}$$

and the corresponding estimates for the sequence r^n; it follows that, for $n \ge 2$,

$$\begin{aligned} & |||s^{n+1}|||^{\Omega,T}_3 + |||r^{n+1}|||^{\Omega,T}_3 \le K^T_2 \, (J_0(3) + 1 \\ & + \left(T + \sqrt{T}\right) \left(|||r^n|||^{\Omega,T}_3 + |||r^{n-1}|||^{\Omega,T}_2 + |||s^n|||^{\Omega,T}_3 + |||s^{n-1}|||^{\Omega,T}_2 \right)) \end{aligned} \tag{3.9}$$

where K^T_1, K^T_2 depend on the L^∞ norm of the data, S^T_∞, C^T_∞, R^T_∞, B^T_∞, the range of the values assumed by F_i, G_i, ψ, φ_i, $i = 1, 2$ and their derivatives, T and Ω, increases with T and it is strictly positive for $T = 0$.

Let $s^1, r^1 \in (W^P(\Omega, T))^2$ and let $\Sigma > 0$ be large such that

$$J_0(3), |||s^1|||^{\Omega,T}_3 \ , |||r^1|||^{\Omega,T}_3, |||s^2|||^{\Omega,T}_3 \ , |||r^2|||^{\Omega,T}_3 < \Sigma \ ;$$

since

$$||| \cdot |||_2 \le ||| \cdot |||_3 + D(\Omega, T) \ ,$$

where $D(\Omega, T)$ is a quantity depending only on Ω and on T, increasing with T, now, it is easy to show that, for suitable small T , the following uniform estimate holds for the sequences s^n, r^n, for $n \ge 3$,

$$|||s^n|||^{\Omega,T}_3 + |||r^n|||^{\Omega,T}_3 \le K^T_3 (\Sigma + 1) \ . \tag{3.10}$$

Now, by compactness techniques, we are able to prove the following existence theorem, in the case of initial data s_0, r_0 belonging respectivelyto the spaces $W^{2,P}(\Omega_1)$ and $W^{2,P}(\Omega_2)$ and c_0, b_0 belonging respecyively to the spaces $W^{2,P+1}(\Omega_1)$ and $W^{2,P+1}(\Omega_2)$.

Theorem 3.1. *Let assumptions a)-c) and (LIB) of Section 1 be satisfied and let* $s_0 \in W^{2,P}(\Omega_1)$, $r_0 \in W^{2,P}(\Omega_2)$ $c_0 \in W^{2,P+1}(\Omega_1)$, $b_0 \in W^{2,P+1}(\Omega_2)$. *Then, there exists a local weak solution* (s, r, c, b) *to problem (1.1)-(1.4) in* $\Omega \times (0, T)$, *for suitable small* T; *moreover* $s, r \in (C([0, T]; W^{2,P}(\Omega_1)) \cap C^1([0, T]; L^P(\Omega_1)))$ *and* $c, b \in (C([0, T]; W^{2,P}(\Omega_2)) \cap C^1([0, T]; L^P(\Omega_2)))$.

Proof. Taking into account that inequalities like (2.15) hold for the functions s^n and r^n, thanks to inequality (3.10) we know that the sequences $\{s^n\}, \{r^n\}, \{c^n\}, \{b^n\}$ are uniformly bounded in the space $C([0, T]; W^{2,P}(\Omega)) \cap C^1([0, T]; L^P(\Omega))$, for suitable small T; then, for such T, some subsequences of $\{s^n\}, \{r^n\}, \{c^n\}, \{b^n\}$ converge weakly* in the space $C([0, T]; W^{2,P}(\Omega)) \cap C^1([0, T]; L^P(\Omega))$ to limit functions, respectively, $s, r, c, b \in C([0, T]; W^{2,P}(\Omega)) \cap C^1([0, T]; L^P(\Omega))$.

For all $z \in C^1_0(\overline{\Omega}_1 \times [0, T))$, the functions of the sequence $\{s^n\}$ satisfy the following problem

(3.11)

$$\begin{aligned}\int_{Q_1^T} \varphi_1(c^n) s^n z_t \; dxdt &= -\int_{\Omega_1} \varphi_1(c_0) s_0 z(x, 0) \; dx + \int_{Q_1^T} \varphi_1(c^n) \nabla s^n \nabla z \; dxdt \\ &- \int_0^T \int_\Gamma \varphi_1(c^n) \psi_1(s^n, r^n) z d\sigma dt \\ &- \int_{Q_1^T} \left(F_1(s^n, c^n) - \varphi_1'(c^n)(G_1(s^n, c^n) - G_1(s^{n-1}, c^n) s^n \right) z \; dxdt\end{aligned}$$

and the functions of the sequence $\{c^n\}$ can be written

$$c^n(x, t) = c_0(x) + \int_0^t G_1(s^{n-1}(x, \tau), c^n(x, \tau)) d\tau \; ; \tag{3.12}$$

similar equalities can be written for r^n and b^n.

Since the sequences s^n, r^n, c^n, b^n are relatively compact in $L^2(Q_1^T)$ and $\nabla s^n, \nabla r^n$ are bounded in the same space, we can pass to the limit for $n \to \infty$, along subsequences, and show that the set (s, c, r, b) is a local weak solution of problem (1.1)-(1.4). □

4. Global existence of weak solutions, stability and uniqueness

In this section, for simplicity of notations, we consider the functions s, r, c, b, solutions to problem (1.1)-(1.4), as extended functions over the set Ω.

We first show that the local solution obtained in Section 3 can be extended over every time interval $[0, T)$. Then we establish a stability theorem, with respect to the data, for weak solutions to problem (1.1)-(1.4), belonging to $(C([0, T); W^{1,P}(\Omega)))$. This result will imply the global existence for solutions $(s, r, c, b) \in (C([0, T); W^{1,P}(\Omega)))^2 \times (C([0, T); W^{1,P+2}(\Omega)))^2$ and the uniqueness.

Theorem 4.1. *Let the assumptions a)-c) and (LIB) of Section 1 be satified, let* $s_0 \in W^{2,P}(\Omega_1)$, $c_0 \in W^{2,P+1}(\Omega_1)$, $r_0 \in W^{2,P}(\Omega_2)$, $b_0 \in W^{2,P+1}(\Omega_2)$ *and let* $[0, T)$ *be the maximal time interval of existence for a weak solution. If* $T < +\infty$, *then the quantities* $|||s|||_P^{\Omega,T}$, $|||r|||_P^{\Omega,T}$ *are bounded.*

Proof. It is possible to fix $\tau > 0$ small enough such that (3.9) implies

$$|||s|||_3^{\Omega,\tau} + |||r|||_3^{\Omega,\tau} \le K_3^T \left(J_0(3) + 1 \right) \; , \tag{4.1}$$

where K_3^T depends on the L^∞ norm of the data, on the a priori bounds S_∞^T, C_∞^T, R_∞^T, B_∞^T, on the range of the values assumed by F_i, G_i, ψ, φ_i, $i = 1, 2$ and by their derivatives, T and Ω; we remark that the choice of τ small depends on these quantities, therefore, in a finite number of time steps of lenght equal to τ, we prove the claim. $\square$

In particular the above theorem shows that, if $[0,T)$ is the maximal time interval of existence for a weak solution and $T < +\infty$ then the $C([0,T];W^{2,P}(\Omega))$ norms of s and r and the $C([0,T];W^{2,P+1}(\Omega))$ norms of c and b are bounded. Then we have the following global existence result.

Theorem 4.2. *Let the assumptions a)-c) and (LIB) of Section 1 be satified and let* $s_0 \in W^{2,P}(\Omega_1)$, $c_0 \in W^{2,P+1}(\Omega_1)$, $r_0 \in W^{2,P}(\Omega_2)$, $b_0 \in W^{2,P+1}(\Omega_2)$. *Then there exists a global weak solution to problem (1.1)-(1.4),* $(s,c,r,b) \in (C([0,T];W^{2,P}(\Omega)) \cap C^1([0,T];L^P(\Omega)))^4$ *for all* $T > 0$.

Now we establish a stability result for solutions to problem (1.1)-(1.4) in the norm

$$|f|_2 := \sup_{[0,T]} \|f(\cdot,x)\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega\times(0,T))} .$$

In the following lemma we claim a stability result for c and b; the proof does not have any differences from the one in [12], since the nonlinear boundary conditions have not direct influence on c and b, so it is omitted.

Lemma 4.1. *Let the assumptions a)-c) and (LIB) of Section 1 be satified, let* $s_0 \in W^{2,P}(\Omega_1)$, $c_0 \in W^{2,P+1}(\Omega_1)$, $r_0 \in W^{2,P}(\Omega_2)$, $b_0 \in W^{2,P+1}(\Omega_2)$, *let* $T > 0$ *and let* $(s_0,c_0,r_0,b_0), (s_0^*,c_0^*,r_0^*,b_0^*)$ *nonnegative data verifying the assumptions (1.4). If there exist solutions* $(s,c,r,b),(s^*,c^*,r^*,b^*) \in (C([0,T];W^{1,P}(\Omega)))^4$ *to problem (1.1), corresponding respectively to initial-boudary data* (s_0,c_0,r_0,b_0), $(s_0^*,c_0^*,r_0^*,b_0^*)$, *then there exists a constant* C, *depending on the data,* Ω_1, Ω_2, P, T, S_∞^T, C_∞^T, R_∞^T, B_∞^T *and the* $C([0,T];W^{1,P}(\Omega))$*- norms of the solutions, such that*

$$|c-c^*|_2^2 \le C\left(\|c_0-c_0^*\|^2_{H^1(\Omega_1)} + T|s-s^*|_2^2\right) ,$$

$$|b-b^*|_2^2 \le C\left(\|b_0-b_0^*\|^2_{H^1(\Omega_2)} + T|r-r^*|_2^2\right) .$$

Theorem 4.3. *Let the assumption a)-d) of Section 1 be satisfied, let* $T > 0$ *and let* $(s_0,c_0,r_0,b_0),(s_0^*,c_0^*,r_0^*,b_0^*)$ *nonnegative data verifying the assumptions (1.4). If there exist solutions* $(s,c,r,b),(s^*,c^*,r^*,b^*) \in (C([0,T];W^{1,P}(\Omega)))^4$ *to problem (1.1), corresponding respectively to initial-boudary data* (s_0,c_0,r_0,b_0) *and* $(s_0^*,c_0^*,r_0^*,b_0^*)$, *then there exists a constant* K, *depending on the data,* Ω_1, Ω_2, P ,T, S_∞^T, C_∞^T, R_∞^T, B_∞^T *and the* $C([0,T];W^{1,P}(\Omega))$*- norms the solutions, such that*

$$\begin{aligned} |s-s^*|_2^2 + |r-r^*|_2^2 \le \quad & K\Big(\|s_0-s_0^*\|^2_{L^2(\Omega_1)} + \|r_0-r_0^*\|^2_{L^2(\Omega_2)} \\ & + \|c_0-c_0^*\|^2_{H^1(\Omega_1)} + \|b_0-b_0^*\|_{H^1(\Omega_2)}\Big) . \end{aligned} \tag{4.2}$$

Proof. We consider the equality (2.6) for $u = s^*$ and for $u = s$; setting $\eta = (s-s^*)z^2$, $\xi = s-s^*$, $\omega = r-r^*$, $y = c-c^*$ and $\theta = b-b^*$, we obtain

$$\begin{aligned} & \int_{Q_1^T} \left(\xi_t \xi z^2 + |\nabla\xi|^2 z^2 + 2z\xi\nabla\xi\cdot\nabla z + (A-A^*)\xi z^2\right) dxdt \\ & \int_0^T \int_{\Omega_1} \left(\psi_s s_{x_n} + \psi_r r_{x_n} - \psi^*_{s^*} s^*_{x_n} - \psi^*_{r^*} r^*_{x_n}\right) \xi z^2 dxdt \\ & + \int_{0_1}^T \int_{\Omega_1} (\psi-\psi^*))\,(\xi_{x_n} z^2 + 2\xi z z_{x_n}) dxdt = 0 , \end{aligned} \tag{4.3}$$

where $\psi = \psi(s, r)$, $\psi* = \psi(s^*, r^*)$ and

$$A = -\frac{1}{\varphi_1(c)}\left(\varphi_1'(c)\nabla c \nabla s + F_1(s,c) - \varphi'(c)G_1(s,c)s\right) \ ,$$

$$A^* = -\frac{1}{\varphi_1(c^*)}\left(\varphi_1'(c^*)\nabla c^* \nabla s^* + F_1(s^*,c^*) - \varphi'(c^*)G_1(s^*,c^*)s^*\right) \ .$$

Then we have

$$\begin{aligned}\int_{\Omega_1}\frac{\xi^2}{2}z^2\Big|_{0_1}^{T} + \int_{Q_1^T}|\nabla\xi|^2 z^2 dxdt \le C\Bigg(\epsilon\int_{Q_1^T}(|\nabla\xi|^2 + |\nabla\omega|^2)z^2 dxdt \\ + \int_{Q_1^T}\left((1+\frac{1}{\epsilon})|\nabla c|^2\xi^2 + \frac{|\nabla y|^2}{\sigma} + \sigma|\nabla s^*|^2\xi^2 + |\nabla s^*|^2 y^2\right)z^2 dxdt \\ + \int_{Q_1^T}\left(\left(1+\frac{1}{\epsilon}\right)(|\xi|^2 + |\omega|^2) + (|\nabla s^*| + |\nabla r^*|)(\xi^2+\omega^2)\right)z^2 dxdt \\ \int_{Q_1^T}\left((\xi^2+\omega^2)zz_{x_n} + \xi^2 zz_t + \frac{1}{\epsilon}\xi^2 z_{x_n}^2\right)dxdt\Bigg)\end{aligned}$$

where $C = C(\|\varphi\|_{C^2}, \|\psi\|_{C^2})$. A similar inequality con be obtained for ω; so we can write

$$\begin{aligned}\int_{\Omega}\frac{\xi^2+\omega^2}{2}z^2\Big|_{0}^{T} + \int_{Q^T}\left(|\nabla\xi|^2 + |\nabla\omega|^2\right)z^2 dxdt \le C\Bigg(\epsilon\int_{Q^T}(|\nabla\xi|^2 + |\nabla\omega|^2)z^2 dxdt \\ + \int_{Q^T}\left((1+\frac{1}{\epsilon})|\nabla c|^2\xi^2 + \frac{|\nabla y|^2}{\sigma} + |\nabla s^*|^2\xi^2 + |\nabla s^*|^2 y^2\right)z^2 dxdt \\ + \int_{Q^T}\left((1+\frac{1}{\epsilon})|\nabla b|^2\omega^2 + \frac{|\nabla\theta|^2}{\sigma} + \sigma|\nabla r^*|^2\omega^2 + |\nabla r^*|^2\theta^2\right)z^2 dxdt \\ + \int_{Q^T}\left(\left(1+\frac{1}{\epsilon}\right)(|\xi|^2 + |\omega|^2) + (|\nabla s^*| + |\nabla r^*|)(\xi^2+\omega^2)\right)z^2 dxdt \\ \int_{Q^T}\left((\xi^2+\omega^2)zz_{x_n} + (\xi^2+\omega^2)zz_t + \frac{1}{\epsilon}(\xi^2+\omega^2)z_{x_n}^2\right)dxdt\Bigg) \ .\end{aligned}$$

By classical methods, for small ϵ, we obtain

(4.4)

$$\begin{aligned}\int_{\Omega}(\xi^2(T) + \omega^2(T))dx + \int_{Q^T}\left(|\nabla\xi|^2 + |\nabla\omega|^2\right)dxdt \le C_1\Bigg(\int_{\Omega_1}\xi_0^2 dx + \int_{\Omega_2}\omega_0^2 dx \\ + \int_{Q^T}\left(\sigma|\nabla s^*|^2\xi^2 + |\nabla c|^2\xi^2 + |\nabla s^*|^2 y^2 + |\nabla b|^2\omega^2 + \sigma|\nabla r^*|^2\omega^2 + |\nabla r^*|^2\theta^2\right)dxdt \\ + \int_{Q^T}(|\nabla s^*| + |\nabla r^*|)(\xi^2+\omega^2)dxdt + \int_{Q^T}(|\xi|^2 + |\omega|^2 + \frac{|\nabla y|^2}{\sigma} + \frac{|\nabla\theta|^2}{\sigma})dxdt\Bigg)\end{aligned}$$

where C_1 depends also on $\|z_i\|_{C^1}$ and Ω_1, Ω_2.

Now we treat the first term in the second integral on the right hand side in the following manner

$$\int_0^t\int_\Omega |\nabla s^*|^2|\xi|^2 dx\, d\tau \le \|\nabla s^*\|^2_{L^{\frac{2p}{p-1},\frac{2q}{q-1}}(Q^T)}\|\xi\|^2_{L^{2p,2q}(Q^T)}$$

with

$$q=1, \quad p=\tfrac{P}{P-2} \quad \text{when } N=3\ (P=3),$$

$$q=\tfrac{P}{2}, \quad p=\tfrac{P}{P-2} \quad \text{when } N=2\ (P>2),$$

$$q=P, \quad p=\infty \quad \text{when } N=1\ (P=2).$$

For these choices of the parameters (p,r), we have the following embedding (see [16])

$$\|\cdot\|^2_{L^{2p,2q}(Q^T)} \le \gamma|\cdot|^2_2\,, \qquad \text{for suitable } \gamma>0.$$

This yields

$$\int_0^t\int_\Omega |\nabla s^*|^2|\xi|^2 \le \gamma\|\nabla s^*\|^2_{L^{P,\frac{2q}{q-1}}(Q^T)}|\xi|^2_2 \tag{4.5}$$

and, arguing as above, we estimate all the terms in the second and the third integral in the right hand side of (4.4); using Proposition 2.1 and Lemma 4.1 and setting $\sigma=\sqrt{T}$, we have

$$|\xi|^2_2+|\omega|^2_2 \le C_3\left(\int_{\Omega_1}|\xi_0|^2\,dx+\int_{\Omega_2}|\omega_0|^2\,dx+\left(T+\sqrt{T}\right)(|\xi|^2_2+|\omega|^2_2)\right.$$
$$\left.+\left(1+\tfrac{1}{\sqrt{T}}\right)\left(\|y_0\|^2_{H^1(\Omega)}+\|\theta_0\|^2_{H^1(\Omega)}\right)\right)$$

where C_3 depends on the data, $P,\Omega,S^T_\infty,C^T_\infty,R^T_\infty,B^T_\infty$ and on the $C([0,T];W^{1,P}(\Omega))$-norms of the solutions.

Proceeding over time intervals of width τ, with τ small in such a way that $(\tau+\sqrt{\tau})C_3\le\frac12$, we obtain the claim. □

We recall that we can bound the $C([0,T];W^{1,P}(\Omega))$-norms of solutions to problem (1.1)-(1.4) by means of T,Ω,S^T_∞, C^T_∞, R^T_∞, B^T_∞, the $W^{2,\frac{P}{2}}(\Omega_1)$-norm of s_0, the $W^{2,\frac{P}{2}}(\Omega_2)$-norm of r_0, the $W^{1,P+2}(\Omega_1)$-norm of c_0 and the $W^{1,P+2}(\Omega_2)$-norm of b_0 (see Lemma 2.3). This fact , together with the previous stability result, imply the main theorem for the existence and uniqueness of weak solutions to problem (1.1)-(1.4).

Theorem 4.4. *Let assumptions a)-c) and (LIB) of Section 1 be satisfied. Then there exists a unique global weak solution to problem (1.1)-(1.4).*

5. Applications to sulphation and chemotaxis

The results obtained in this paper can be applied to several reaction-diffusion systems proposed as models for chemical and biological processes. In the following we present two specific examples.

Sulphation phenomena. The following nonlinear system of parabolic equations describes the evolution of the chemical reaction of sulphur dioxide with the surface of calcium carbonate stones

$$\begin{cases} \partial_t(\varphi(c)s) &= \operatorname{div}(\varphi(c)\nabla s)-\varphi(c)cs,\\ \partial_t c &= -\varphi(c)cs, \end{cases} \tag{5.1}$$

for $(x,t)\in\Omega\times[0,T]$ $(T>0,\Omega\subset\mathbb{R}^N)$.

Here s stands for the porous concentration of SO_2, namely the concentration taken with respect to the volume of the pores, c for the local density of $CaCO_3$ and the function $\varphi(c)$ is the porosity. This model has been introduced in [2] to describe the transformation in time of $CaCO_3$ (calcium carbonate) stones under the chemical aggression due to the action of SO_2 (sulphur dioxide). This reaction converts the calcium carbonate on the surface of a stone, in a thin crust of calcium sulphate (gypsum). Global existence and uniqueness results and analysis of the macroscopic behavior in time of the crust of calcium sulphate are obtained in [10, 11] in the case of one space dimension and in [12] in the case of several space dimensions. The interested reader can look at [2, 1, 3] and the comprehensive book [9] for more details about the chemical background as well as for other related references. One of the main features of the model is the fact that the porosity function φ is assumed to depend on the local density of the calcite c, actually as a linear function $\varphi(c) = A + Bc$, which is strictly positive on the interval $[0, \|c_0\|_\infty]$.

Here we consider the transmission problem

$$
\begin{cases}
s_t = \Delta s + \frac{\varphi_1'(c)}{\varphi_1(c)} \nabla s \cdot \nabla c - \lambda_1 cs(1 - \varphi_1'(c)s) \ , & \\
 & x \in \Omega_1 \ , t > 0 \ , \lambda_1 > 0 \ , \\
\partial_t c = -\lambda_1 \varphi_1(c) cs \ , & \\
r_t = \Delta r + \frac{\varphi_2'(b)}{\varphi_2(b)} \nabla r \cdot \nabla b - \lambda_2 br(1 - \varphi_1'(b)r) \ , & \\
 & x \in \Omega_2 \ , t > 0 \ \lambda_2 > 0 \ , \\
\partial_t b = -\lambda_2 \varphi_2(b) br \ , &
\end{cases}
\tag{5.2}
$$

complemented with the initial and boundary conditions

$$
\begin{aligned}
& s(x,0) = s_0(x) \ , \ c(x,0) = c_0(x) \ , \quad x \in \Omega_1 \\
& r(x,0) = r_0(x) \ , \ b(x,0) = b_0(x) \ , \quad x \in \Omega_2 \\
& \varphi_1(c) \tfrac{\partial s}{\partial n_1} = r - s \ , \qquad \varphi_2(b) \tfrac{\partial r}{\partial n_2} = s - r \ , \qquad (x,t) \in \Gamma \times (0,T) \quad , \\
& \tfrac{\partial s}{\partial n_1} = 0 \qquad (x,t) \in \partial\Omega_1 \setminus \Gamma \times (0,T) \ , \\
& \tfrac{\partial r}{\partial n_2} = 0 \qquad (x,t) \in \partial\Omega_2 \setminus \Gamma \times (0,T) \ ,
\end{aligned}
\tag{5.3}
$$

where the initial data are nonnegative functions.

It is easy to see that assumptions a)-c) of Section 2 are satisfied. As regard to assumption *(LIB)* we first observe that the right hand sides of the ordinary differential equations are nonpositive; moreover, $c = 0$, respectively $b = 0$, are subsolutions, for all $s, r \geq 0$. So, we immediately obtain the bounds

$$0 < c \leq \|c_0\|_\infty \ , \qquad 0 \leq b \leq \|b_0\|_\infty .$$

This means that there are two strictly positive constants $\varphi_m < \varphi_M$, such that

$$\varphi_m \leq \varphi_1(c), \varphi_2(b) \leq \varphi_M, \qquad \text{for all } c \in [0, \|c_0\|_\infty], b \in [0, \|b_0\|_\infty].$$

In order to obtain bounds for s and r, we are going to prove the following proposition. Let $Z := max\{\|s_0\|_\infty, \|r_0\|_\infty\}$ and let assume that

$$\text{if some } \varphi_i' > 0 \text{ then } Z < \min\left\{ \frac{1}{\varphi_i'} \text{ for } i = 1, 2 \text{ such that } \varphi_i' > 0 \right\} \tag{5.4}$$

Proposition 5.1. *Let assume (5.4) and let r, c be smooth functions such that*

$$0 \leq r(x,t) \leq Z \ , \quad 0 \leq c(x,t) \leq \|c_0\|_\infty \ \forall (x,t) \in \overline{\Omega} \times [0,T] \ .$$

Then the solution s to the problem

$$
(5.5)\qquad \begin{cases}
Ls = s_t - \Delta s - \frac{\varphi'(c)}{\varphi(c)} \nabla s \cdot \nabla c + \lambda c s (1 - \varphi(c) s) = 0, \\[2ex]
\varphi_1(c) \frac{\partial s}{\partial n} = \psi(s, r) = r - s \qquad (x,t) \in \Gamma \times (0,T)\ , \\[2ex]
\frac{\partial s}{\partial n_1} = 0 \qquad (x,t) \in \partial\Omega_1 \setminus \Gamma \times (0,T)\ , \\[2ex]
s(x,0) = s_0 \geq 0,
\end{cases}
$$

for smooth inital data, verifies the following bounds

$$0 \leq s(x,t) \leq Z\ , \forall (x,t) \in \overline{\Omega} \times [0,T]\ .$$

Proof. It is sufficient to observe that $\underline{s} = 0$ is a subsolution and $\overline{s} = Z$ is a supersolution, i.e.

$$L\underline{s} \leq 0\ ,$$

$$\varphi_1(c) \frac{\partial \underline{s}}{\partial n_1} \leq \psi(0, r) \text{ in } \Gamma \times (0,T)\ , \quad \varphi_1(c) \frac{\partial \underline{s}}{\partial n_1} \leq 0 \text{ in } \partial\Omega_1 \setminus \Gamma \times (0,T)\ ,$$

$$\underline{s}(x,0) \leq s_0(x) \text{ in } \Omega_1$$

and

$$L\overline{s} \geq 0\ ,$$

$$\varphi_1(c) \frac{\partial \overline{s}}{\partial n_1} \geq \psi(Z, r) \text{ in } \Gamma \times (0,T)\ , \quad \varphi_1(c) \frac{\partial \overline{s}}{\partial n_1} \geq 0 \text{ in } \partial\Omega_1 \setminus \Gamma \times (0,T)\ ,$$

$$\overline{s}(x,0) \geq s_0(x) \text{ in } \Omega_1$$

and to apply comparison theorems. □

The same result, obviously holds for the function r; then assumption *(LIB)* in the Introduction is satified by system (5.2).

As a consequence of the above considerations, the results of the present paper provide global existence and uniqueness result for weak solutions to system (5.2) when the data satisfy (1.4) and, in the case of some increasing φ_i, under the further smallness assumption on the data (5.4).

We remark that the result applies also in the case of Dirichlet conditions on $\partial\Omega_1 \setminus \Gamma$ and $\partial\Omega_2 \setminus \Gamma$, since Proposition 5.1 holds, provided that Z is larger than the prescribed data on the boundaries.

Another interesting problem is the system (5.2) where the Neumann conditions on Γ are substituited by the following ones, coming from the Kedem-Katchalsky equations applied to biological problems [15, 4],

$$(5.6)\qquad \varphi_1(c) \frac{\partial s}{\partial n_1} = \psi_{\nu,\mu}(s, r)\ ,\ \varphi_2(b) \frac{\partial r}{\partial n_2} = -\psi_{\nu,\mu}(s, r)\ ,$$

$$(5.7)\qquad \psi_{\nu\mu}(s, r) = -\left((1 - \mu)(s - r) + \frac{s - r}{\log \frac{s}{r}} (\nu - \mu(s - r)) \right)\ ,$$

where $0 < \mu < 1$ and $\nu > 0$; moreover we assume $s_0, r_0|_\Gamma > 0$. Also in this case it is possible to show that, for suitable values of μ, that assumptions a)-c) and *(LIB)* in the Introduction are satisfied in such a way the global existence and uniqueness theorem holds; we have only to prove the a priory L^∞ bounds for s, r.

Proposition 5.2. *Let assume (5.4), let r, c be smooth functions such that*

$$0 \leq r(x,t) \leq Z \ , \quad r(x,t)|_\Gamma > 0 \quad , 0 \leq c(x,t) \leq \|c_0\|_\infty \ \forall (x,t) \in \overline{\Omega}_1 \times [0,T]$$

and let $\frac{1-\mu}{\mu} > Z$. Then the solution s to the problem

$$\left\{ \begin{array}{l} Ls = s_t - \Delta s - \frac{\varphi'(c)}{\varphi(c)} \nabla s \cdot \nabla c + \lambda c s (1 - \varphi(c) s) = 0, \\ \\ \varphi_1(c) \frac{\partial s}{\partial n} = \psi_{\nu\mu}(s,r) \qquad (x,t) \in \Gamma \times (0,T) \ , \\ \\ \frac{\partial s}{\partial n_1} = 0 \qquad (x,t) \in \partial\Omega_1 \setminus \Gamma \times (0,T) \ , \\ \\ s(x,0) = s_0(x) > 0, \end{array} \right. \tag{5.8}$$

for smooth inital data, verifies the following bounds

$$0 \leq s(x,t) \leq Z \ \forall (x,t) \in \overline{\Omega}_1 \times [0,T] \ , \ s|_\Gamma > 0.$$

Proof. Let $\epsilon > 0$ small. It is easy to prove, by using the equation, that assuming the existence of $(\overline{x}, \overline{t}) \in \Omega_1 \times (0,T]$ such that $s(\overline{x}, \overline{t}) = -\epsilon$ and $s(x,t) > -\epsilon$ for all $x \in \Omega$, $t < \overline{t}$ leads to a contraddiction; also, this situation cannot occur over $\partial\Omega_1 \setminus \Gamma$. On the other hand, for $r > 0$,

$$\lim_{s \to 0^+} \psi_{\nu\mu}(s,r) > 0 \ ,$$

thus, till $s \geq 0$ in $\Omega_1 \cup \partial\Omega_1 \setminus \Gamma$, it cannot vanish over Γ and the lower bound is proven.

In similar way as above, assuming the existence of $(\overline{x}, \overline{t}) \in \Omega_1 \times (0,T]$ such that $s(\overline{x}, \overline{t}) = Z + \epsilon$ and $s(x,t) < Z + \epsilon$ for all $x \in \Omega_1$, $t < \overline{t}$, leads to a contraddiction; also, this situation cannot occur over $\partial\Omega_1 \setminus \Gamma$.

In order to investigate the behaviour over Γ, we rewrite $\psi_{\nu\mu}$ as follows, using the variable $y = \frac{s}{r} - 1$,

$$\psi_{\nu\mu}(s,r) = -(s-r) \left((1-\mu) + \frac{\nu}{\log(1+y)} - \mu s H(y) \right) , H(y) := \frac{y}{(y+1)\log(1+y)} \ ;$$

notice that, if $s = Z + \epsilon$, then $y > 0$ and, if $y \geq 0$, then $H(y) \leq 1$. These considerations imply that, under the position

$$\frac{1-\mu}{\mu} > Z \ ,$$

s cannot assume the value $Z + \epsilon$ over Γ until $s(x,t) \leq Z + \epsilon$ in $\Omega_1 \cup \partial\Omega_1 \setminus \Gamma$; thus the upper bound is proven,too. ☐

The same result holds for the function r, then the assumptions a)-c) and *(LIB)* in the Introduction are satisfied by system (5.2) with boundary conditions given by (5.6), (5.7), when the data satisfy (1.4) and, in the case of some increasing φ_i, the smallness assumption on the data. Thus, global existence and uniqueness result holds.

Chemotaxis phenomena. A second example of reaction-diffusion system where our results apply, is given by the following class of Keller-Segel type models of chemotaxis

$$\left\{ \begin{array}{ll} \partial_t u & = \mu \Delta u - \nabla \cdot (u \chi(c) \nabla c) + f(u,c) \ , \\ \\ \partial_t c & = g(u,c) \ , \end{array} \right. \qquad (x,t) \in \Omega \times (0,T) \ , \tag{5.9}$$

where u represents the density of some motile living species and c represents the concentration of chemical species. The coefficient $\mu > 0$ is the motility coefficient which here is assumed to be constant and the term $u\chi(c)$ is the chemotactic sensitivity function which here is assumed to be linear in the species u. The function χ is usually assumed nonnegative and non increasing. This class of systems was largely studied when the second equation contains an additional linear diffusion term (see [14] for a survey of results).

Global existence results for system (5.9), with $f(s, c) = 0$ and for particular expressions for g and χ can be found in [5, 22, 12].

Introduce a function $\varphi(c)$ related to the function $\chi(c)$ by the equality

$$\mu\varphi'(c) = \varphi(c)\chi(c);$$

using the new unknown $s = \frac{u}{\varphi(c)}$, and setting $F(s, c) = f(\varphi(c)s, c)$ and $G(s, c) = g(\varphi(c)s, c)$, we obtain the system

$$\left\{ \begin{array}{ll} \partial_t(\varphi(c)s) & = \mu \operatorname{div}(\varphi(c)\nabla s) + F(s, c)\ , \\ \qquad\quad \partial_t c & = G(s, c)\ , \end{array} \right. \qquad (x, t) \in \Omega \times (0, T) \tag{5.10}$$

which belongs to the class studied in this paper, provided that the functions φ, G, F satisfy the assumptions in Section 2; notice that, by definition, φ is nonnegative and then the same holds for $\varphi'(c)$.

We are going to treat the following particular case of transmission problem in chemotaxis

(5.11)

$$\left\{ \begin{array}{ll} \partial_t s = \Delta s + \frac{\varphi_1'(c)\nabla s\cdot\nabla c}{\varphi_1(c)} - \frac{\varphi_1'(c)}{\varphi_1(c)} s(\alpha_1\varphi_1(c)s - \beta_1 c)\ , & \\ & (x, t) \in \Omega_1 \times (0, T)\ , \\ \partial_t c = \alpha_1\varphi_1(c)s - \beta_1 c\ , & \\ \partial_t r = \Delta r + \frac{\varphi_2'\nabla r\cdot\nabla b}{\varphi_2(b)} - \frac{\varphi_2'(b)}{\varphi_2(b)} r(\alpha_2\varphi_2(b)r - \beta_2 b)\ , & \\ & (x, t) \in \Omega_2 \times (0, T)\ ; \\ \partial_t b = \alpha_2\varphi_2(b)r - \beta_2 b\ , & \end{array} \right.$$

we are assuming that the chemotctic sensitivity function is strongly influenced by underlying substrate.

The system is complemented with the initial and the boundary conditions

$$\begin{array}{l} s(x, 0) = s_0(x)\ ,\ c(x, 0) = c_0(x)\ ,\ x \in \Omega_1 \\ r(x, 0) = r_0(x)\ ,\ b(x, 0) = b_0(x)\ ,\ x \in \Omega_2 \\ \varphi_1(c)\frac{\partial s}{\partial n_1} = \psi(s, r)\ ,\ \varphi_2(b)\frac{\partial r}{\partial n_2} = -\psi(s, r)\ , \qquad (x, t) \in \Gamma \times (0, T)\ , \\ \frac{\partial s}{\partial n_1} = 0\ , \qquad (x, t) \in \partial\Omega_1 \setminus \Gamma \times (0, T)\ , \\ \frac{\partial r}{\partial n_2} = 0\ , \qquad (x, t) \in \partial\Omega_2 \setminus \Gamma \times (0, T)\ , \end{array} \tag{5.12}$$

where the initial data are nonnegative functions. Also in this case, global existence and uniqueness of solutions to system (5.11) follow, provided that the assumptions in the Introduction are verified. In particular we will show that a priori L^∞ bounds in assumption *(LIB)* are verified for some classes of functions φ_i, when the boundary conditions are given by $\psi(s, r) = r - s$ or by (5.6) and(5.7).

1) φ_i satisfies assumptions of Section 2 and, for $i = 1, 2$,

$$\sup_{z\in[0,+\infty)} \frac{z}{\varphi_i(z)} \leq L \ , \qquad z\varphi_i(z) \leq K_2 z^2 + K_1 \ , \qquad K_1, K_2 > 0 \ \ , z \geq 0 \ . \tag{5.13}$$

Let $Z := max\left\{\|s_0\|_\infty, \|r_0\|_\infty, L\frac{\beta_1}{\alpha_1}, L\frac{\beta_2}{\alpha_2}\right\}$.

Proposition 5.3. *Let r be a given smooth function such that*

$$0 \leq r(x,t) \leq Z \quad \forall (x,t) \in \overline{\Omega}_2 \times [0,T]$$

and let conditions (5.13) be satisfied; then

a) *if $0 \leq s \leq Z$ then the solution c to the problem (5.11)-(5.12), for all $(x,t) \in \overline{\Omega}_1 \times [0,T]$, verifies*

$$c_0(x)e^{-\beta_1 t} \leq c(x,t) \leq \frac{K_2 c_0(x)^2 + K_1}{K_2} e^{2Z\alpha_1 K_2 t} - \frac{K_1}{K_2} =: C_\infty^T \ ;$$

b) *if $0 \leq c(x,t) \leq C_\infty^T$ for all $(x,t) \in \overline{\Omega}_1 \times [0,T]$ then the solution s to the problem (5.11)-(5.12), with $\psi(s,r) = r - s$, verifies*

$$0 \leq s(x,t) \leq Z \quad \forall (x,t) \in \overline{\Omega}_1 \times [0,T] \ ;$$

if ψ is given by(5.6) and (5.7) the same lower bound holds while the upper one holds in the case $\frac{1-\mu}{\mu} > Z$.

Proof. The claim a) easily follows by applying comparison results to the differential equation satisfied by c, also using the second assumption in (5.13).

The second claim can be proved by the same techniques used in [12] and the same method used in the proof of the previous two propositions, to treat the boundary conditions.

□

Proceeding as in Proposition 5.3, under the same assumptions, we prove the bounds for b and r, for all $(x,t) \in \overline{\Omega}_2 \times [0,T]$,

$$b_0(x)e^{-\beta_2 t} \leq b(x,t) \leq \frac{K_2 b_0(x)^2 + K_1}{K_2} e^{2Z\alpha_2 K_2 t} - \frac{K_1}{K_2} =: B_\infty^T \ ,$$

$$0 \leq r(x,t) \leq Z \ .$$

The class of functions φ_i we are dealing with, contains obviously linear functions, strictly positive for $c \in \overline{\mathbb{R}^+}$, which correspond to chemiotactic functions of the form $\chi(c) = \frac{A}{Ac+B}$, for suitable coefficients A, B.

Notice that, if $c_0(x) \geq \tilde{c} > 0$, then it is sufficient to ask that, for all $T > 0$,

$$\sup_{c\in[\tilde{c}e^{-\beta T},+\infty)} \frac{c}{\varphi(c)} \leq L(T).$$

2) φ_i satisfies assumptions of Section 2 and

$$0 < \varphi_m \leq \varphi_i(z) \leq \varphi_M \quad \text{ for } z \in \overline{\mathbb{R}^+} \ , \quad i = 1, 2. \tag{5.14}$$

In this case, the same techniques used in [12] and approaching the boundary conditions as in the previous propositions, leads to the following a priori L^∞ bounds

$$0 \leq s(x,t) \leq Z \ \forall (x,t) \in \overline{\Omega}_1 \times [0,T] \ , \ 0 \leq r(x,t) \leq Z \ \forall (x,t) \in \overline{\Omega}_2 \times [0,T],$$

where $Z := \max\left\{\|s_0\|_{L^\infty(\Omega)}, \|r_0\|_{L^\infty(\Omega)}, 2\frac{\beta_1\|c_0\|_\infty}{\alpha_1\varphi_m}, 2\frac{\beta_2\|b_0\|_\infty}{\alpha_2\varphi_m}\right\}$, and

$$c_0(x)e^{-\beta_1 t} \leq c(x,t) \leq \frac{\alpha_1}{\beta_1}\varphi_M Z + \|c_0\|_\infty \ , \quad \forall (x,t) \in \overline{\Omega} \times [0,T] \ ,$$

$$b_0(x)e^{-\beta_2 t} \leq b(x,t) \leq \frac{\alpha_2}{\beta_2}\varphi_M Z + \|b_0\|_\infty \ , \quad \forall (x,t) \in \overline{\Omega} \times [0,T] \ .$$

The choice $\chi(c) = \frac{1}{(c+\delta)^\alpha}$, $\alpha > 1$, $\delta > 0$, leads to $\varphi(c) = e^{-\frac{1}{(\alpha-1)(c+\delta)^{\alpha-1}}}$ which belongs to the class of functions verifying (5.14). As in the previous example, if $c_0(x) \geq \tilde{c} > 0$, then c remain strictly positive , in such a way that, if $\varphi(0) = 0$ (e.g. $\varphi(c) = e^{-\frac{1}{(\alpha-1)c^{\alpha-1}}}$, $\alpha > 1$), then the above argument can be applied over every interval $[0,T]$, with $\varphi_m = \inf_{[\tilde{c}e^{-\beta T}, +\infty]} \varphi(c)$. Finally let us remark that for system (5.11), the global existence and uniqueness result applies also in the case of Dirichlet conditions on $\partial\Omega_1 \setminus \Gamma$ and $\partial\Omega_2 \setminus \Gamma$.

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